# Naked Singularities in Self-Similar Spherical Gravitational Collapse

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We present general-relativistic solutions of self-similar collapse of an adiabatic perfect fluid. We show that if the equation of state is soft enough ($\Gamma - 1 \ll 1$), a naked singularity forms. The singularity resembles the shell-focusing naked singularities that arise in dust collapse. This solution increases significantly the range of matter fields that should be ruled out in order that the cosmic-censorship hypothesis will hold.

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The cosmic-censorship conjecture[1] is generally accepted as the most important current open question in classical general relativity.[1,2] This conjecture suggests that a space-time singularity which develops in the future of a set of regular initial data cannot be seen by an observer (strong version) or at least by an external observer (weak version). If the weak version of the cosmic-censorship conjecture is true, predictability is saved, at least in the region external to the event horizon, even when singularities form. If the strong version is true, predictability is saved everywhere.

By now, several counterexamples to this conjecture have been found.[2-5] Most of these examples are based, however, on collapse of pressureless matter and it is generally believed that these naked singularities will be engulfed by an event horizon when pressure is introduced. We have investigated[6] the self-similar[7] spherical collapse of a perfect fluid with an adiabatic equation of state $p=(\Gamma-1)\rho$. We find that when $\Gamma-1\ll 1$, a naked singularity may appear. This solution increases significantly the range of matter fields that should be ruled out in order that the cosmic-censorship hypothesis will hold.

We describe the collapse by the total-energy density $\rho = d/t^2$, the velocity $u^r$, and the metric functions $g_{rr}=e^\lambda$ and $g_{tt}=-e^\nu$. We use a radial area coordinate $r$ and an orthogonal time coordinate $t$. We look for a solution in which $u^r$, $\lambda$, $\nu$, and $d$ are functions of $x \equiv r/|t|$. The spherical self-similar relativistic collapse equations are

$$8\pi\Gamma d u_t u^r = (e^{-\lambda}-1)/x + 8\pi d(1+\Gamma u^r u_r)x, \tag{1a}$$

$$8\pi d(1+\Gamma u_t u^t) = -e^{-\lambda}(1/x^2 + \nu'/x) + 1/x^2, \tag{1b}$$

$$8\pi\Gamma d u_t u^r = e^{-\lambda}\lambda', \tag{1c}$$

$$[u^r d' + u^t(2d + xd')] + \Gamma d u^r/x + e^{-(\lambda+\nu)/2}[(e^{(\lambda+\nu)/2}u^r)' + x(e^{(\lambda+\nu)/2}u^t)'] = 0, \tag{1d}$$

where the prime denotes a derivative with respect to $x$. Regularity at the origin requires $u^r(0)=\lambda(0)=0$. We make an arbitrary choice of $\nu(0)=0$.

The central density, $\rho(0)=d_0/t^2$, diverges at $t=0$ if $d_0 \neq 0$ (as we will see later $\lim_{x\to\infty} d = 0$ and only the central density diverges). This singularity is a basic feature of the solution and it does not reflect any singularity in the solution of Eqs. (1a)–(1d). In the rest of the paper we demonstrate the existence of regular solutions to Eqs. (1a)–(1d) and we describe solutions in which null geodesics originating at the singularity at (0,0) reach infinity.

The solution is characterized by two parameters, $\Gamma$ and $d_0$. For a given choice of these parameters we integrate Eqs. (1a)–(1d) numerically, from $x=0$ towards $x=\infty$. We consider only $\Gamma < \Gamma_b \approx 1.015$. (If $\Gamma > \Gamma_b$ the relativistic Penston-Larson solutions, that we discuss below, contain trapped surfaces and a black hole forms before $t=0$.)

The self-similar solution passes through a sonic point $x_s$ [a test particle on the world line $r=|t|x_s$ moves at the speed of sound $c_s=(\Gamma-1)^{1/2}$ relative to the fluid]. The solution is generally not regular at $x_s$, and the first or one of the higher derivatives of $d$ and $u^r$ diverges there. For a given $\Gamma$ there exist, however, a discrete set of values of $d_0$ for which the solution is regular.[8] One of these regular solutions, the general-relativistic equivalent of the Penston-Larson Newtonian solution,[8] seems to be the simplest candidate for a naked-singular solution. From now on, we shall examine this specific solution.

Figure 1 displays a numerical solution of Eqs. (1a)–(1d) for $\Gamma=1.01$ and $d_0=0.1144$, which corresponds to the special regular solution. To understand the nature of our example it is sufficient to consider the solution near the origin and at infinity. Near the origin the solution describes an almost homogeneous ($d \approx d_0 - d_2 x^2$), uniform ($u^r \approx -2x/3\Gamma$), Newtonian ($2m/r = 1 - g_{rr}^{-1}$

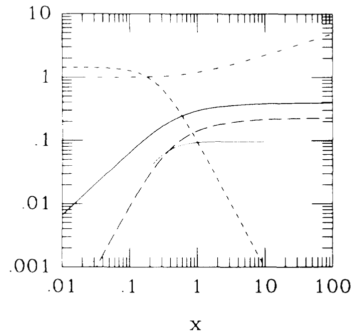


FIG. 1. Self-similar collapse expressed in Schwarzschild coordinates for $t<0$ ($\Gamma=1.01$ and $d_0=0.1144$): $|u^r|$ (solid line), $4\pi x^2 d=4\pi p r^2$ (dotted line), $4\pi d$ (short-dashed line), $2m/r$ (long-dashed line), and $|g_{tt}|$ (dash-dotted line). Note that $2m/r<1$ for all $x$ values.

$\approx 8\pi x^2 d_0/3 \ll 1$) collapse. The solution goes over asymptotically to an isothermal [$d\approx d_\infty x^{-2}$, $2m/r \approx (2m/r)_\infty$], constant-velocity ($u^r \approx -u_\infty$) infall. Just like the shell-focusing singularities,[2,3] this singularity has a Newtonian character, $2m/r<1$ for all $x$ and hence for all $r$ (for $t<0$). The system remains almost Newtonian and a black hole does not appear before $t=0$, i.e., until the singularity is formed.

The complete self-similar solution contains, however, an infinite total mass and is not asymptotically flat. To obtain an asymptotically flat solution we introduce a cutoff in the density profile at $r_{c0}=|t_0|x_c$. The density drops smoothly to zero at $r>r_{c0}$. The cutoff destroys the self-similarity for $r>r_c(t)$. The inner region, $r<r_c(t)$, retains, however, its self-similar structure. If $r_{c0}$ is large enough, the cutoff will not influence the region near the singularity. In particular,

$$x_p=\exp[\{\nu(x_p)-\lambda(x_p)\}/2]$$

defines an ingoing radial null geodesic of the form $r=x_p|t|$. World lines of the form $r=x|t|$ are timelike, null, or spacelike, for $x<x_p$, $x=x_p$, or $x>x_p$, respectively. If we impose the condition $x_c>x_p$, perturbations introduced at $x_c>x_p$ by the cutoff will not influence the singularity at ($r=0, t=0$) or its immediate nearby region.

For large $x$, $\nu \approx \alpha \ln(x)$ {where $\alpha=\Gamma d_\infty/(1-2m/r)_\infty][1+2u_\infty^2/(1-2m/r)_\infty]$}, and the time coordinate is singular at $t=0$. To avoid this singularity we transform to comoving coordinates $R,T(u^R=0)$. We map ($r=0,t=0$) to ($R=0,T=0$). The line $T=0$ is, however, earlier than the line $t=0$ (see Fig. 2). On $T=0$, $\Psi \propto -\ln(T)$ (where $g_{TT}=-e^{\Psi}$) and the coordinate $T$ is singular. We transform at $t_t<0$ and $T_t>0$ and we bypass both coordinate singularities.

An alternative transformation is $\tilde{t}=t^{1-\alpha}$. With this transformation we bypass the singularity at $t=0$ and we describe the solution using different ($r,t$) coordinate patches.[6] It is, however, more convenient to switch to the ($R,T$) coordinates. The region where ($r,t$) and ($R,T$) overlap does not have any special significance. However, comparison of the numerical solutions using the ($r,t$) coordinates (and $\tilde{t}=t^{1-\alpha}$ around $t=0$) and using the ($R,T$) coordinates provided us with an estimate of accuracy of the numerical calculations. Both calculations agree for at least four significant figures. These numerical errors are sufficiently small so that they do not change the nature of the solution and our conclusions.

To form a self-similar solution, we define $y=R/T$ and we look for a solution where $g_{TT}=-e^{\Psi}$, $g_{RR}=e^{\Lambda}$, $D=\rho T^2$, and $\tilde{r}=r/T$ are functions of $y$ only. The comoving self-similar collapse equations are

$$y\tilde{r}''-\frac{(\Gamma-1)}{\Gamma}(\tilde{r}-y\tilde{r}')\frac{D'}{D}+y\tilde{r}'\left[\Gamma\frac{D'}{D}+\frac{2\tilde{r}'}{\tilde{r}}+\frac{2(\Gamma-1)}{y}\right]=0, \tag{2a}$$

$$e^{-\Psi}\left[2y^2\tilde{r}\tilde{r}''+(\tilde{r}-y\tilde{r}')^2-\frac{2(\Gamma-1)y\tilde{r}}{\Gamma}(\tilde{r}-y\tilde{r}')\frac{D'}{D}\right]+e^{-\Lambda}\left[(\tilde{r}')^2-\frac{2(\Gamma-1)\tilde{r}\tilde{r}'}{\Gamma}\frac{D'}{D}\right]=-8\pi(\Gamma-1)D\tilde{r}^2-1, \tag{2b}$$

$$e^{-\Psi}=(4\pi D)^{2(\Gamma-1)/\Gamma}, \tag{2c}$$

$$e^{-\Lambda}=(4\pi D)^{2\Gamma}\tilde{r}^4 y^{4(\Gamma-1)}, \tag{2d}$$

where the prime denotes derivative work with respect to $y$.

We integrate numerically the comoving equations, from the vicinity of $T=0$ (i.e., a large value of $y$), towards $y=0$ (see Fig. 3). The solution diverges at $y_s$: $D\propto(y-y_s)^{-\Gamma/(2-\Gamma)}$, $\nu\approx[(2\Gamma-2)/(2-\Gamma)]\ln(y-y_s)$, $\lambda\approx-[2/(6-3\Gamma)]\ln(y-y_s)$, and $\tilde{r}\propto(y-y_s)^{2/(6-3\Gamma)}$. The spherical fluid shells crash into a central $r=0$ singularity on the world line $y_s$. For $T>0$, the singularity is "massive" and it is surrounded by an apparent horizon. The mass of the singularity and the size of the apparent horizon (see Figs. 2 and 3) grow linearly with the comoving time $T$:

$$\lim_{y\to y_s}\left(\frac{2m}{T}\right)=\lim_{y\to y_s}\left(\frac{2m}{r}\tilde{r}\right)>0.$$

The singularity is spacelike for $T>0$ and no photons can escape from it later than $T=0$. The singularity and its casual structure resemble the shell-focusing singularities found in the Tolman-Bondi dust solutions (see, in particular, Fig. 1 of Ref. 3). A careful analysis reveals that

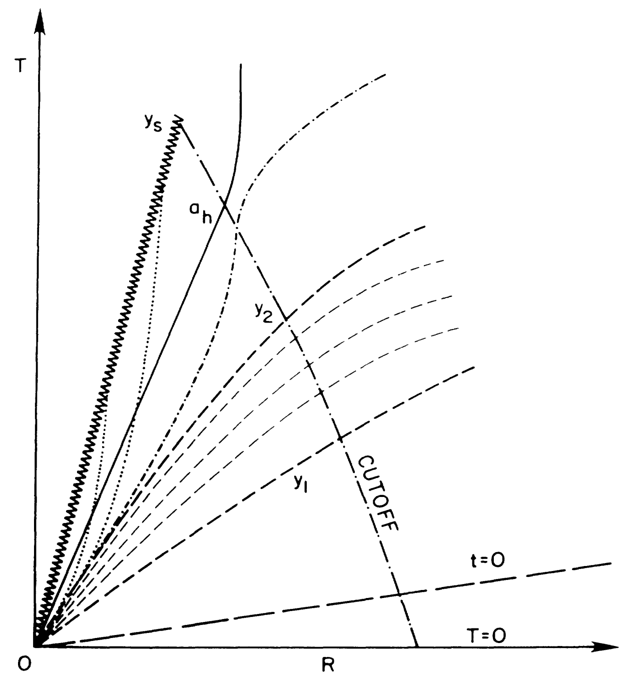


FIG. 2. A schematic space-time diagram of the collapse in comoving coordinates. The singularity at $y_s$ is represented by a sawtoothlike line. The apparent horizon is denoted by $a_h$. The cutoff is denoted by a long-dash–dotted line. Dashed lines denote null geodesics that are between $y_1$ and $y_2$ and escape to infinity. Dotted lines denote null geodesics that are between $y_2$ and $y_s$ and fall back into the singularity. The short-dash–dotted line denotes a geodesic that is between $y_2$ and $y_s$ and would have fallen into the singularity, but it escapes to infinity because of the cutoff.

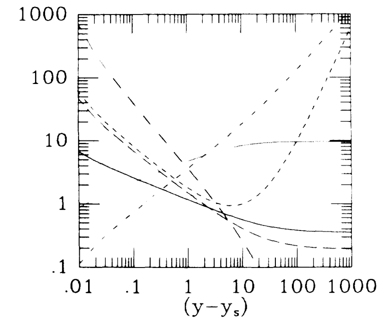


FIG. 3. Self-similar collapse in comoving coordinates for $T>0$ ($\Gamma=1.01$ and $d_0=0.1144$): $|u^r|$ (solid line), $400\pi\rho r^2=400\pi\tilde{r}^2D$ (dotted line), $4000\pi D$ (long-dash–dotted line), $2m/r$ (long-dashed line), $\tilde{r}=r/T$ (short-dash–dotted line), and $F(y)=y^2g_{RR}(y)/|g_{TT}(y)|$ (short-dashed line).

the singularity at (0,0) extends along a null interval. The Ricci scalar diverges on the singularity and unlike the shell-focusing singularities[4] this singularity is a strong-curvature singularity.[9] The massive singularity that develops later resembles the central Schwarzschild singularity.

A simple radially outgoing null geodesic of the form $R=\text{const}\times T$ [from the singularity at (0,0) to infinity] exists if $F(y)\equiv y^2\exp(\Lambda-\Psi)=1$ for some $y>y_s$. For $\Gamma<\Gamma_c\approx 1.0105$, there are two solutions $y_1$ and $y_2$ $[F(y_1)=F(y_2)=1]$. *The singularity at $(R=0,T=0)$ is naked for* $\Gamma<\Gamma_c$. The line $R=y_1T$ is a Cauchy horizon. We can, however, complete the solution by analytic continuation. In this solution there exists an infinite set of null geodesics that emerges from the singularity. These geodesics are divided into two families: geodesics that reach infinity, and geodesics that are trapped by the expanding apparent horizon and fall into the central singularity (see Fig. 2). In the complete self-similar solution $R=y_2T$ behaves like an event horizon and separates the two families. With a cutoff the space-time is asymptotically flat and the outgoing geodesics reach null infinity. The null geodesics diverge faster (compared to a self-similar region) in the exterior region (see Fig. 2). As a result, some of the null geodesics of the second family, which emerge from the singularity between $y_2$ and $y_s$, escape infinity, instead of falling into the singularity (see Fig. 2). The event horizon is located between $R=y_2T$ and the apparent horizon.

The red shift from a source located at the center ($r=0$ and $t<0$) diverges like $[r_c(t=0)/|t|]^\alpha$ as $t\to 0$. Therefore, a distant observer will see the singularity only if it has an infinite luminosity. However, the dynamics of causal future of the singularity (the domain $y<y_1$) depends on boundary conditions at the singularity, and is not predictable from the initial data on $t_0$. The solution that we have described is based on an analytic extension of the solution from $y>y_1$ to $y<y_1$, which is equivalent to the assumption that *no perturbations are coming out from the singularity.* Clearly, one can imagine boundary conditions that will lead to different solutions at $y<y_1$ which will influence an external observer.

We have described a family of general-relativistic solutions for self-similar spherical collapse of an adiabatic perfect fluid that include naked singularities and provide a counterexample to the cosmic-censorship hypothesis. Our example resembles the shell-focusing naked singularities that appear in pressureless collapse in spite of the fact that our matter field has nonvanishing and unbound pressure.

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[1]R. Penrose, Riv. Nuovo Cimento Soc. Ital. Fis. **1**, 252 (1969); for a recent review, see W. Israel, Found. Phys. **14**,

1049 (1984).

[2]D. M. Eardley, in *Gravitation and Astrophysics*, edited by B. Carter and J. Hartle (Plenum, New York, 1987).

[3]D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979); D. M. Eardley, Gen. Relativ. Gravitation **10**, 1033 (1979).

[4]D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984); R. P. A. C. Newman, Classical and Quantum Gravity **3**, 527 (1986).

[5]W. A. Hiscock, L. G. Williams, and D. M. Eardley, Phys. Rev. D **26**, 751 (1982); B. Waugh and K. Lake, Phys. Rev. D **34**, 2978 (1986).

[6]A. Ori and T. Piran, Gen. Relativ. Gravitation (to be published), and to be published.

[7]For a discussion of general relativistic self-similar behavior, see K. P. Stanyukovich, O. Sharshekeev, and V. Ts. Gurovich, Dokl. Akad. Nauk SSSR **165**, 510 (1965) [Sov. Phys. Dokl. **10**, 1030 (1966)]; M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971); D. M. Eardley, Commun. Math. Phys. **37**, 287 (1974); O. I. Bogoyavlenskii, Zh. Eksp. Teor. Fiz. **73**, 1201 (1977) [Sov. Phys. JETP **46**, 633 (1977)]; G. V. Bicknell and R. N. Henriksen, Astrophys. J. **225**, 237 (1978).

[8]A similar behavior appears in Newtonian self-similar collapse, see M. V. Penston, Mon. Not. Roy. Astron. Soc. **144**, 449 (1969); R. B. Larson, Mon. Not. Roy. Astron. Soc. **145**, 271 (1969); C. Hunter, Astrophys. J. **218**, 834 (1977), for a discussion of self-similar isothermal collapse; A. Yahil, Astrophys. J. **265**, 1047 (1983), for a discussion of the polytropic collapse.

[9]K. Lake, to be published.