Optical Analog of a Kicked Quantum Oscillator

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(Received 11 August 1987)

Light propagation in a discrete waveguide is the classical-optics analog of the quantum dynamics of a kicked oscillator. A specific model is introduced which shows subdiffusive beam spreading in the ray-optical limit. At finite wavelength, the beam is localized in both position and momentum space through wave interference effects. It is argued that the localization phenomenon is accessible to a laboratory experiment using conventional laser systems.

PACS numbers: 03.65.Sq, 05.45.+b, 42.20.Cc

The quantum dynamics of classically chaotic Hamiltonian systems has attracted great interest in recent years.[1,2] Much work has been devoted to the study of the kicked quantum rotator first introduced by Casati *et al.*[3–9] This model displays the remarkable phenomenon of quantum localization in momentum (energy) space, in contrast to the unbounded diffusive growth of energy in the classical system. Grempel, Prange, and Fishman established a close formal relationship to Anderson localization in a random potential.[7] Energy localization has been predicted for more realistic systems, such as a diatomic molecule[9] and a highly excited hydrogen atom[10] in a microwave field. However, so far the effect has not been observed in any experiment.

Here I propose to study localization in an *optical* analog of a kicked quantum oscillator. It is shown that a beam of light passing through a discrete anharmonic waveguide is localized in both position and (optical) momentum space. The width of the localized beam scales with the wavelength λ of the light, which plays the role of Planck's constant. In the ray-optical (classical) limit $\lambda \to 0$, the beam spreads subdiffusively. By variation of the wavelength and the dimensions of the waveguide, both the classical (delocalized) and the quantum (localized) regimes are accessible to standard optical technology. Besides providing the first experimental confirmation of the phenomenon, such an experiment would contribute to a better understanding of the interference effects which lead to localization. In this sense, my approach is similar in spirit to some recent studies of Anderson localization in classical wave problems.[11,12]

I consider a chain of identical optical elements, aligned along the z axis with equal spacings z_0. The elements are assumed to be purely refractive and ideally thin. For simplicity, I take them to be cylindrical, with a refractive index variation $V(q)$ only in one transverse direction q. For a quadratic function $V(q)=q^2/2f$, this defines a standard lens waveguide.[13] The propagation of a light ray along the chain is described, in the Hamiltonian formulation of geometrical optics, by phase-space variables (q,p), where $p=n_0(dq/dz)$ is proportional to the slope of the ray. The free-space refractive index, n_0, will be set equal to unity. In the paraxial approximation, $|p| \ll 1$, the ray equations are then Hamilton's equations corresponding to the optical Hamiltonian

$$\begin{aligned} H(q,p,z) &= \tfrac{1}{2}p^2+\Delta n(q,z), \\ \Delta n(q,z) &= V(q)\sum_{l=-\infty}^{\infty}\delta(z-lz_0). \end{aligned} \tag{1}$$

Obviously, if z is regarded as time, this is the Hamiltonian of a kicked oscillator. There is a one-to-one correspondence between paraxial optics and nonrelativistic mechanics both in the full wave-optical (quantum) theory and in the ray-optical (classical) limit.[13] In wave-optical terms, a scalar wave field $\psi(q,z)$ propagates through the waveguide according to the paraxial wave equation

$$i\lambdabar\,\partial\psi/\partial z = -\tfrac{1}{2}\lambdabar^2\partial_q^2\psi+\Delta n\psi. \tag{2}$$

This is Schrödinger's equation with Planck's constant h replaced by the vacuum wavelength λ, $\lambdabar=\lambda/2\pi$. The wave field $\psi_n(q)$ at $z=nz_0$ is mapped to the wave field $\psi_{n+1}(q)$ at $z=(n+1)z_0$ by the one-step propagator

$$\hat{U}=\exp[-i(\hat{p}^2/2\lambdabar)z_0]\exp[-iV(q)/\lambdabar]. \tag{3}$$

Here $\hat{p}=-i\lambdabar\partial_q$ is the optical momentum operator. Equation (3) defines a quantum map of the type first studied by Berry *et al.*[14]

I choose for $V(q)$ a piecewise parabolic double-well potential:

$$V(q)=\begin{cases}\{(q+2q_0)^2-q_0^2\}/2f & \text{for } q<-q_0,\\ \{q_0^2-q^2\}/2f & \text{for } q^2<q_0^2,\\ \{(q-2q_0)^2-q_0^2\}/2f & \text{for } q>q_0.\end{cases} \tag{4}$$

This choice is motivated by both practical and theoretical considerations. Practically, a refractive index variation of the type (4) could be realized by joining together two convex (for $q^2>q_0^2$) and one concave (for $q^2<q_0^2$) cylindrical lenses with focal length f in such a way that the pieces fit smoothly at $q=\pm q_0$. Theoretically, the

potential (4) is convenient because the corresponding area-preserving map [Eq. (5) below] shows a slow (subdiffusive) escape rate of trajectories to infinity, even in the unbounded chaotic regime. This is important in order to have a well-defined classical long-time behavior which can then be compared to the quantum dynamics.

This model is oversimplified in several respects. Firstly, I neglect any effects from the boundaries of the optical elements: The potential $V(q)$ in (4) extends along the whole q axis. A real experiment would further suffer from reflection losses, which may have to be compensated for by introduction of amplification into the chain. Finally, in an experimental realization it may be more convenient to work in a ring geometry rather than with a linear chain of elements. A well-defined number of passes could then be achieved by use of a short laser pulse. All of these points would have to be taken into account for a detailed comparison with future experiments. In the present work my goal is to identify the main effect and to estimate the range of parameters where it could be observed. Therefore, I will consider only the simple model given by (1) and (4).

Let me briefly describe the ray-optical limit $\lambda \to 0$. The ray equations corresponding to (4) are given by a piecewise linear, continuous map of the (q,p) plane. I introduce scaled variables $x = q/q_0$ and $y = p/p_0$, where $p_0 = q_0/z_0$ is a typical slope in the waveguide. Then the map is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \gamma & 1+\gamma \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad |x_n + y_n| < 1, \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \pm \begin{pmatrix} 0 \\ 2\gamma \end{pmatrix}, \quad x_n + y_n \begin{cases} > 1, \\ < -1, \end{cases} \tag{5}$$

and $\gamma = z_0/f$. If $0 < \gamma < 4$, the map is elliptic in the regions $|x+y| > 1$. Points iterate around the stable fixed points $(\pm 2, 0)$ at angular frequency $\omega = \arccos(1 - \gamma/2)$. In the region $|x+y| < 1$, the map is hyperbolic with an unstable fixed point at the origin. In the neighborhood of the origin the motion is chaotic because of homoclinic crossings of the stable and unstable manifolds.[15] The global behavior then depends on whether there exists an invariant curve around the origin, which confines the stochastic motion. The presence of such a curve is determined by the parameter γ in a complicated way, which is only partly understood.[15] In particular, the usual notion of a stochasticity threshold associated with the breaking of the last Kol'ogorov-Arnol'd-Moser curve does not apply to the piecewise linear map (5). If no invariant curve exists, we find numerically that an ensemble of trajectories launched close to the origin spreads both in position and momentum according to a power law

$$\langle q^2 \rangle_n, \langle p^2 \rangle_n \propto n^{\alpha}. \tag{6}$$

Here the brackets denote ensemble averaging and n is the number of iterations along the chain. I will often refer to n as "time." The exponent α in (6) is less than unity, $\alpha \cong 0.67$. This *subdiffusive* behavior can be understood as follows. I write the time evolution of some coordinate, say p, as a generalized diffusion law with a time-dependent diffusion constant $D(n)$:

$$\langle p^2 \rangle_n = D(n) n. \tag{7}$$

To estimate $D(n)$, I note that the motion in the elliptic regions $|x+y| > 1$ is completely regular. Only trajectories within $|x+y| < 1$ contribute to the spreading of the ensemble. Therefore, $D(n)$ is proportional to the fraction of the ensemble with coordinates in the hyperbolic region. If we further assume uniform spreading, then this fraction decreases as $\langle p^2 \rangle_n^{-1/2}$, since $|x+y| < 1$ defines a quasi one-dimensional strip in the phase plane. Inserting into (7) we obtain $\alpha = \frac{2}{3}$, which is in good agreement with numerical calculations at various (diffusive) values of γ.[15]

Now I ask how the diffusive behavior of the ray map (5) is affected by interference effects due to a finite wavelength λ. The analogy with the kicked rotator leads us to expect some kind of localization phenomenon. However, my model is different from the rotator in several respects. In particular, the spectrum of the unperturbed system [$V=0$ in (3)] is continuous rather than discrete. As a consequence, the main results on the dynamics of the kicked quantum rotator—the occurrence of quantum resonances[4] and the mapping to the Anderson problem[7]—cannot be carried over to my case. Therefore I undertook some exploratory numerical calculations. Before discussing the results, let me add a remark concerning the natural observables in the waveguide. In the classical limit, the evolution of $\langle q^2 \rangle_n$ and $\langle p^2 \rangle_n$ was used as a measure for phase-space diffusion. Now the ensemble average is replaced by the quantum-mechanical expectation value in the state $\psi_n(q)$. Then $\langle q^2 \rangle$ is just the mean square width of the beam. To see the physical significance of the "kinetic energy" $\langle p^2 \rangle$ in the optical context, I write

$$\langle p^2 \rangle = -\lambda\!\!\!^{-}{}^2 \langle \psi | \partial_q^2 | \psi \rangle = \lambda\!\!\!^{-}{}^2 \int dp\, p^2 |\hat{\psi}(p)|^2, \tag{8}$$

where $\hat{\psi}(p)$ is the Fourier transform of the wave field $\psi(q)$. If $p = q/\lambda\!\!\!^{-} f_0$, where f_0 is the focal length of a focusing lens, then $|\hat{\psi}(p)|^2$ gives the intensity distribution in the focal plane of the lens.[13] Thus $\langle p^2 \rangle^{1/2}$ is proportional to the focal spot size, which is commonly used as a measure for laser beam quality.

Some of the results for $\langle p^2 \rangle_n$ are shown in Fig. 1. The uppermost curve is the classical power law (7). The curves at finite λ were obtained by iterating the wave map (3) numerically, with fast Fourier transform on a grid of 2^{15} points. The propagator $\hat{U}$ depends on two di-

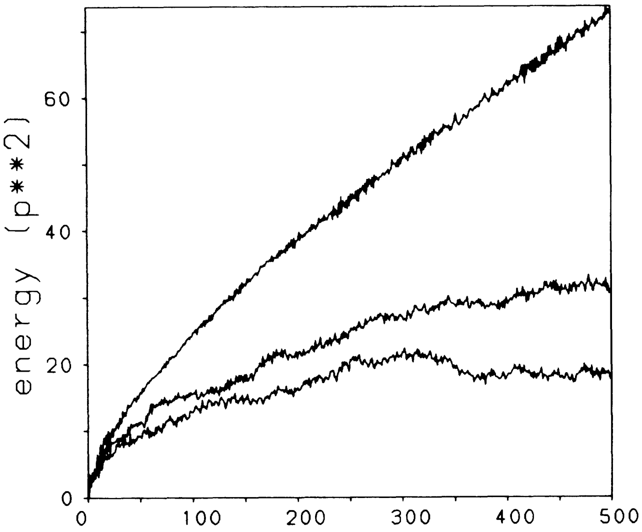

FIG. 1. Scaled energy $\langle (p/p_0)^2 \rangle_n$ of the quantum oscillator as a function of the number of kicks. The uppermost curve is the classical limit, $\lambda = 0$. The wavelength for the middle curve is $\lambda = 6.5$ μm, for the lower curve $\lambda = 10.6$ μm.

mensionless parameters

$$\sigma = \lambda z_0 / q_0^2, \quad \tau = q_0^2 / \lambda f. \tag{9}$$

The classical parameter is $\gamma = \sigma\tau$. The dimensions of the waveguide were fixed by setting the width of the central defocusing lens $2q_0 = 1$ cm, the focal length $f = 50$ cm, and the distance between elements $z_0 = 100$ cm. Then $\gamma = 2$, and the ray dynamics is subdiffusive as described above. The initial condition was a Gaussian centered on the optical axis, $\psi_0(q) \cong \exp[-(q/\Delta)^2]$ with $\Delta = q_0/2$. The wavelength was varied between 1.3 and 26 μm. For short wavelengths ($\lambda \leq 3.4$ μm) and short times ($n \leq 100$) the classical and the quantum results for $\langle p^2 \rangle_n$ are almost indistinguishable. This confirms that my calculations correctly reproduce the ray-optical limit $\lambda \to 0$. For $\lambda = 10.6$ μm, which is the wavelength of a CO_2 laser, the picture is drastically different (lowest curve in Fig. 1). At $n = 100$, the energy differs from its classical value by a factor of 2. After 300 iterations, the growth saturates and $\langle p^2 \rangle_n$ starts to oscillate around some mean value, which I call the saturation energy $E_s(\lambda)$. The width of the beam, $\langle q^2 \rangle_n$, similarly saturates at some value $W_s(\lambda)$. The wave functions appear to be exponentially localized in both position and momentum space.[15] In terms of the scaled variables (x,y) introduced above (5), E_s and W_s^2 are equal to within 30%, i.e., we have the following approximate relation:

$$E_s \cong (p_0/q_0)^2 W_s^2 = (W_s/z_0)^2. \tag{10}$$

This indicates that localization is uniform in phase space. The width of the localized beam is typically a few times the width of the central defocusing lens. For $\lambda = 10.6$ μm, I found $W_s = 3.8 q_0 = 1.9$ cm. The case $\lambda = 6.5$ μm (second curve in Fig. 1) is intermediate. The growth of energy is much slower than for $\lambda = 0$, but no saturation was observed up to $n = 1000$. Thus we see a clearcut transition between classical and quantum behavior over a few hundred iterations, which occurs in the range of wavelengths 3.4 μm $\leq \lambda \leq$ 10.6 μm. This range is clearly accessible to standard laser systems.

For a quantitative estimate of the wavelength dependence of the saturation energy, I adopt a simple heuristic argument given by Chirikov, Izrailev, and Shepelyansky, and Casati *et al.*, in the context of the kicked rotator.[5,8] I extended their derivation to the case of subdiffusive classical energy growth, $\langle p^2 \rangle_n \propto n^\alpha$, and obtained

$$E_s \propto \lambda^{-2\alpha/(2-\alpha)}. \tag{11}$$

With $\alpha = \frac{2}{3}$ from (7), this gives $E_s \propto 1/\lambda$. According to (10), the width of the localized beam should then scale as $W_s(\lambda) \propto \lambda^{-1/2}$. My data for various values of λ between 10.6 and 26 μm are consistent with these relations. Of course, this does not exclude that a transition to a delocalized regime with unbounded energy growth may take place at some smaller, but finite, wavelength. Further numerical work is required to clarify this question.

In summary, I have shown that diffusive beam spreading in an anharmonic optical waveguide is localized through wave interference effects. By analogy with mechanics, this provides a new example of energy localization in a driven quantum oscillator, which has the great advantage of being directly realizable in a laboratory experiment.

It is a pleasure to thank H. Spohn for many valuable suggestions on the manuscript, and K. J. Witte for very useful discussions.

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