Bistability Driven by Weakly Colored Gaussian Noise: The Fokker-Planck Boundary Layer and Mean First-Passage Times

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We develop a singular perturbation approach to the problem of first-passage times for non-Markovian processes driven by Gaussian colored noise near the white-noise limit. In particular we treat the problem of overdamped tunneling in a bistable quartic potential in the presence of Gaussian noise of short correlation time τ. The correct treatment of the absorbing boundary yields a lowest-order correction proportional to $\tau^{1/2}$ with a proportionality constant involving the Milne extrapolation length for the Fokker-Planck equation, given in terms of the Riemann ζ function.

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The question of time scales in nonlinear stochastic-dynamic systems has come under intense scrutiny in recent years. Among these various time scales are the correlation time,[1] the time scales associated with the spectrum of an appropriate time-development operator,[2] and mean first-passage times (MFPT's).[3] In many applications the underlying stochastic process can be taken as a Markov process—for example, a Markov diffusion process defined by a Fokker-Planck equation (FPE)—but there is also interest in the time scales involved with non-Markovian processes.[3,4] Non-Markovian processes, often nonlinear dynamical systems perturbed by colored-noise forces, serve as models for the effect of pump noise in lasers,[5] the influence of environmental fluctuations on chemical and biological systems,[6] fluid-turbulence studies,[7] and a regularization of quantum field theories.[8]

Many recent theoretical, numerical, and experimental studies have concentrated on the fundamental question of the lifetime of a metastable state perturbed by a colored Gaussian noise, and one particular model has received the greatest attention: overdamped tunneling in a bistable quartic potential in the presence of weakly colored Gaussian noise.[9] The model is defined by the stochastic differential equation

$$dx/xt = f(x) + u(t), \tag{1}$$

where $f(x)$ is the force due to the bistable potential $V(x)$:

$$f(x) = x - x^3 = -\frac{dV(x)}{dx}, \qquad V(x) = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2, \tag{2}$$

and the noise $u(t)$ is Gaussian with correlation time τ and intensity D:

$$\langle u(t)u(s)\rangle = (D/\tau)\exp(-|t-s|/\tau). \tag{3}$$

These equations are written in a dimensionless form where time is measured in units of the deterministic relaxation-time scale in the wells of the quartic potential, and the state variable and noise amplitude have been rescaled in terms of the parameters in the potential. In the limit $\tau \to 0$, the state variable $x(t)$ satisfies a stochastic differential equation driven by a Gaussian white noise, and it is then a Markov diffusion process whose transition density obeys a FPE. Current studies of the MFPT focus on the transition from one well of the potential (e.g., at $x=1$) to the unstable point ($x=0$), and for the most part these studies have centered on the development of an "effective" FPE for the time development of the probability distribution of $x(t)$.[9,10] There are presently four such effective FPE's in the literature which have recently been summarized and compared in Ref. 10.

Rather than searching for an effective Markov process to describe $x(t)$, we consider the joint stochastic process consisting of the state variable, x, and the Gaussian colored-noise variable, an Ornstein-Uhlenbeck process. The joint process *is* Markovian, and thus the first-passage problem may be formulated according to the standard techniques for systems whose transition density obeys a FPE. The MFPT problem cannot be solved exactly as in the case of a one-variable Markov diffusion process, however, and so we develop a systematic perturbation expansion of the MFPT in powers of the correlation time of the Gaussian noise.

Equations (1)-(3) are rewritten in terms of the simultaneous stochastic differential equations

$$dx/dt = f(x) + \epsilon^{-1}\sigma z(t), \qquad dz/dt = -\epsilon^{-2}z + \sqrt{2}\epsilon^{-1}\xi(t), \tag{4}$$

where $\xi(t)$ is a δ-correlated Gaussian white noise. The dimensionless correlation time of the Gaussian process $z(t)$ is $\tau=\epsilon^2$, and the dimensionless intensity of the noise is $\sigma=\sqrt{D}$. The FPE for the time development of the probability density of the joint process is

$$\partial_t \rho(x,z,t) = [\epsilon^{-2}\partial_z(z+\partial_z) - \epsilon^{-1}\sigma z\,\partial_x - \partial_x f(x)]\rho(x,z,t). \tag{5}$$

We will treat ϵ as a small parameter, thus considering the situation where the memory-time scale of the fluctuations is much less than the deterministic relaxation-time scale imposed by the quartic potential.

The MFPT problem is usually expressed in terms of the adjoint FPE, but it can also be formulated in terms of a stationary density associated with the stochastic dynamics of the system.[11] The physical idea is to inject independent (i.e., noninteracting) particles into the system at a fixed rate with annihilation at the absorbing boundaries, and let the total density build up to a stationary state. The average total number of particles present in the stationary state is exactly the product of the rate of introduction and the MFPT to the boundaries. Thus, for the dynamics at hand, the MFPT of the variable x from the well at $x=1$ to the unstable point at $x=0$ may be obtained by solution of the equation

$$[\epsilon^{-2}\partial_z(z+\partial_z) - \epsilon^{-1}\sigma z\,\partial_x - \partial_x f(x)]G(x,z) = -\delta(x-1)\rho_0(z) \tag{6}$$

for $G(x,z)$, the total average stationary density of particles present when they are introduced at unit rate at $x=1$ with the stationary distribution of the noise variable z:

$$\rho_0(z) = (2\pi)^{-1/2}\exp(-z^2/2). \tag{7}$$

By introducing the particles with the stationary distribution of the noise variable, we assume that the noise is always in its stationary state. The MFPT is given by the total average number of resident particles:

$$T = \int_{-\infty}^{\infty} dz \int_0^{\infty} dx\, G(x,z). \tag{8}$$

The statement of the problem is not yet complete, though, because the partial differential equation for $G(x,z)$ [Eq. (6)] requires the specification of boundary conditions. The boundary conditions on the density are usually taken to be vanishing on the absorbing boundary, but, in the case at hand, the absorbing boundary is *not* correctly specified by the demand that $G(0,z)=0$ for all values of the noise variable z. The absorbing boundary for this problem is a boundary in the phase space spanned by x and z, and the appropriate boundary condition is $G(0,z)=0$ only for $z>0$. Physically, this means that there are no particles *entering* the region $x>0$, i.e., the x component of the incoming probability current $\{[\epsilon^{-1}\sigma z + f(x)]G(x,z)$ for $z>0$ at $x=0\}$ vanishes on the absorbing boundary. The other boundary conditions are that the density vanishes as either $x\to\infty$ or $z\to\pm\infty$.

The perturbative treatment of the problem proceeds according to the techniques described by Horsthemke and Lefever[6,12] for weakly colored Gaussian noise forces, and by Blankenship and Papanicolaou[13] for more general "fast" noises. This technique is essentially that used to reduce the Klein-Kramers equation (describing Brownian motion in phase space) to the Smoluchowski equation (for Brownian motion in configuration space) near the high-friction limit.[14] Writing Eq. (6) as

$$(\epsilon^{-2}L_0 + \epsilon^{-1}L_1 + L_2)G(x,z) = -\delta(x-1)\rho_0(z), \tag{9}$$

where the differential operators L_i are

$$L_0 = \partial_z(z+\partial_z), \quad L_1 = -\sigma z\,\partial_x, \quad L_2 = -\partial_x f(x), \tag{10}$$

we insert the *Ansatz*

$$G(x,z) = G_0(x,z) + \epsilon G_1(x,z) + \epsilon^2 G_2(x,z) + \cdots \tag{11}$$

into Eq. (9) and collect terms according to powers of ϵ. The coefficients of the expansion obey the equations

$$L_0 G_n(x,z) = -L_1 G_{n-1}(x,z) - L_2 G_{n-2}(x,z) - \delta_{n,2}\delta(x-1)\rho_0(z), \tag{12}$$

where $G_n(x,z)\equiv 0$ for $n<0$. At this point it is convenient to introduce the eigenfunctions and the spectrum of L_0:

$$L_0\rho_n(z) = -n\rho_n(z), \quad \rho_n(z) = H_n(z)\rho_0(z), \tag{13}$$

where the Hermite polynomials $H_n(z)$ are fully described by Abramowitz and Stegun.[15] The Hermite polynomials and the eigenfunctions satisfy the useful recursion relation

$$z\rho_n(z) = \rho_{n+1}(z) + \rho_{n-1}(z). \tag{14}$$

For $n=0$ and $n=1$, Eq. (12) yields

$$G_0(x,z) = \rho_0(z)r_0(x), \quad G_1(x,z) = \rho_0(z)r_1(x) - \rho_1(z)\sigma\partial_x r_0(x), \tag{15}$$

where $r_0(x)$ and $r_1(x)$ are yet to be determined.

With use of Eqs. (15) and the recursion relation Eq. (14), the equation for $G_2(x,z)$ is rewritten

$$L_0G_2(x,z)=\rho_0(z)[-\delta(x-1)-\sigma^2\partial_x^2 r_0(x)+\partial_x f(x)r_0(x)]+\rho_1(z)\sigma\partial_x r_1(x)-\rho_2(z)\sigma^2\partial_x^2 r_0(x), \tag{16}$$

and the fundamental feature of this singular perturbation theory comes into play. The operator L_0 has a 0 eigenvalue (with eigenfunction ρ_0) and is thus not generally invertible. It is invertible, however, on the subspace of functions of z spanned by the $\rho_n(z)$'s for $n\ge 1$. Hence, in order that we be able to solve Eq. (16) for G_2, the coefficient of ρ_0 must vanish. This yields a partial differential equation for $r_0(x)$:

$$-\partial_x f(x)r_0(x)+\sigma^2\partial_x^2 r_0(x)=-\delta(x-1). \tag{17}$$

In general, the partial differential equation satisfied by the "reduced" density $r_n(x)$, appearing as the coefficient of $\rho_0(z)$ in the solution for G_n, will arise as an integrability condition for the partial diffusion equation for G_{n+2} obtained from the expansion Eq. (12). The functions $r_n(x)$ are called the reduced densities because the marginal (or reduced) density for the x variable, denoted $r(x)$ and obtained by the integration of the full solution $G(x,z)$ over z, has an expansion in powers of ϵ of the form $r(x)=r_0(x)+\epsilon r_n(x)+\epsilon^2 r_n(x)+\cdots$.

The singular perturbation expansion continues by our expressing the solution for G_2 in terms of the functions $\rho_n(z)$ and $r_n(x)$ $(n=0,1,2)$. The equation for G_3 is then rearranged with use of the recursion relation Eq. (14), into an expansion involving the variable z only through the functions $\rho_n(z)$ $(n=0,1,2,3)$. The integrability condition, that the coefficient of ρ_0 must vanish, leads to the equation for $r_1(x)$,

$$-\partial_x f(x)r_1(x)+\sigma^2\partial_x^2 r_1(x)=0. \tag{18}$$

The technique should now be clear, and we quote the result for the next order in the reduced density:

$$[-\partial_x f(x)+\sigma^2\partial_x^2]r_2(x) =-\sigma^2\partial_x[-\partial_x f(x)+\sigma^2\partial_x^2]\partial_x r_0(x). \tag{19}$$

Before we set out to solve Eqs. (17)–(19), it is necessary to specify the boundary conditions for the reduced densities. The boundary conditions for the absorbing boundary are really imposed on the current in the full phase space, and they do not reduce directly to some boundary conditions on the reduced densities. In order to derive the implications of the full boundary conditions on the reduced densities, one must solve the so-called "boundary layer" problem. In essence, this involves our exactly solving the force-free analog of Eq. (6) near the absorbing boundary $x=0$ in the full phase space. This exact solution has been derived, but the details will be presented elsewhere.[16] The result for the problem at hand is simply summarized, through order ϵ^2, by the imposition of the reduced boundary condition

$$r(0)=\epsilon\sigma\lambda_M r'(0) \tag{20}$$

on the total reduced density $r(x)$, where λ_M is the "Milne extrapolation length" for the FPE given in terms of the Riemann ζ function by

$$\lambda_M=-\zeta(\tfrac{1}{2})=1.460354\ldots. \tag{21}$$

The Reimann ζ function appears infrequently in physics problems, but we remark in passing that this is not the only connection between the ζ function and stochastic processes.[17] We also mention that while reduced boundary conditions like those above were conjectured by Chandrasekhar[18] in 1943, the absorbing boundary conditions in phase space were written down by Wang and Uhlenbeck[19] in 1945, and there have been some very accurate numerical[20] and analytical[21] approximations, the exact Milne extrapolation length has eluded computation until very recently (see Ref. 16 and the independently derived result of Marshall and Watson[22]).

We are now in a position to solve the problem at hand. The solutions to Eqs. (17)–(19) [with the boundary conditions implied by Eq. (20)] are

$$r_0(x)=\begin{cases}\sigma^{-2}\exp[-V(x)/\sigma^2]\int_0^1 dx'\exp[V(x')/\sigma^2], & x\ge 1,\\ \sigma^{-2}\exp[-V(x)/\sigma^2]\int_0^x dx'\exp[V(x')/\sigma^2], & 0\le x\le 1,\end{cases} \tag{22a}$$

$$r_1(x)=\lambda_M\sigma^{-1}\exp[-V(x)/\sigma^2], \tag{22b}$$

$$r_2(x)=\delta(x-1)+V''(x)r_0(x)-\int_0^x dx' r_0(x')V'(x')V''(x')\partial_{x'}g(x,x'), \tag{22c}$$

where $g(x,x')$ in Eq. (22c) is the Green's function for the white-noise Fokker-Planck operator with vanishing boundary conditions at $x=0$ [i.e., $g(x,x')$ is just $r_0(x)$ given by Eq. (22a) above with the 1's in the first integral and the limits on x replaced by x'].

The MFPT is evaluated to second order in ϵ from Eq. (8) and the expressions above, where we note that only the coefficients of ρ_0 in the G_n's contribute to the MFPT. The complete expression is somewhat cumbersome, but it can be greatly simplified if we restrict ourselves to the small-noise-amplitude regime ($\sigma^2\ll 1$). Then the integrals are easily evaluated by the method of steepest descent, and the MFPT is

$$T=\frac{\pi}{\sqrt{2}}\exp\left(\frac{1}{4\sigma^2}\right)\left[1+\left(\frac{2}{\pi}\right)^{1/2}\lambda_M\epsilon+\tfrac{3}{2}\epsilon^2+\cdots\right],\quad \sigma^2\ll 1. \quad (23)$$

In the above we have neglected terms of order ϵ^2 with exponentially small coefficients [i.e., terms containing a factor of $\exp(-1/4\sigma^2)$ inside the bracket in Eq. (23)] and terms with $O(\sigma^2)$ coefficients.

The general technique for our systematically finding corrections to quantities near the white-noise limit presented in this Letter is applicable to many problems. For example, in the more general multiplicative noise problem, where σ^2 is a function of x, the expansion presented here can be similarly carried out. With absorbing boundaries located at deterministic critical points, the reduced boundary condition Eq. (20) carries through exactly. The approach presented here also has some advantages over the effective FPE approach[9,10] mentioned earlier. An important problem with all of the effective FPE's is their failure to yield any small-correlation-time corrections for free processes, i.e., when $f(x)\equiv 0$ in Eq. (1). The singular perturbation method developed here does not suffer from this malady. In fact, the MFPT for free process driven by a colored Gaussian noise, computed by the methods presented here, agrees qualitatively with the exact solutions for some non-Gaussian colored noises like the dichotomous Markov process.[16,23]

Finally, it is important to stress that the effective FPE's never pick up the leading-order corrections to the MFPT proportional to $\tau^{1/2}$ which result from the correct treatment of the reduced boundary conditions at the absorbing boundary. [In the context of the problem at hand, we note that the correction of order $\epsilon^2=\tau$ derived from most of the effective FPE's agrees[10] with that in Eq. (23).] This new result calls for new numerical and analog experiments on the MFPT for bistable flows in the small-noise-correlation-time regime. Neglect of the $O(\tau^{1/2})$ term can be misleading if the next higher-order correction proportional to τ is negative so that one might conclude that small correlations *decrease* a MFPT (for an example, see Doering[24]). The reduced boundary conditions imply that the MFPT from any point to a deterministic critical point always has a *positive* lowest-order correction: Very small Gaussian noise correlations *increase* transition times to deterministic critical points in one-dimensional systems.

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