Integrable Ponderomotive System: Cavitons are Solitons

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A new set of integrable soliton equations describe caviton formation in a plasma. The equations are solvable by the inverse scattering transform. The time evolution of the scattering data is obtained and it is found that, unlike other integrable equations where the magnitude of the reflection coefficient was always a constant of the motion, here it may grow and/or decay in time. One physical manifestation of the growth of the reflection coefficient is the growth of density ripples in front of a microwave source. On the other hand, when the decay conditions dominate, one is left with only cavitons, which are the solitons for these equations.

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I have discovered a new integrable system describing "caviton" formation.[1-4] Cavitons are important in the study of long-wavelength Langmuir turbulence.[1-8] Most studies are based on the Zakharov equations[9] which describe the interactions of ion-acoustic density waves with electron (Langmuir) plasma waves. Langmuir waves oscillating at a frequency just above the electron plasma frequency become modulationally unstable and collapse into a region of strong localized electrostatic electric fields, accompanied by a density depression (whence the name cavitons).

The Zakharov equations are considered to be nonintegrable, except in two limits: the nonlinear Schrödinger equation[6] and the Yajima-Oikawa equations.[10] However, several other limits possess solitary waves[11-13] and some of these solitary waves may well be solitons.

I shall here discuss a generalization of the Karpman equations [Eqs. (28) and (29) of Ref. 11]. This generalization is

$$\partial_x^2 E + (\alpha^2 - q)E = 0, \tag{1}$$

$$\partial_\tau q + \partial_x \int_{-\infty}^{\infty} d\alpha\, E^* E(\alpha, x, \tau) = 0. \tag{2}$$

In the above, $E(\alpha,x,\tau)$ is an rf electric field, α is the (scaled) frequency above the plasma frequency, and q is the fractional change in the plasma density. The integral over α in (2) represents the cumulative ponderomotive effect due to a spectrum of incoherent and/or turbulent sources. In the case of a δ-function spectrum, we would recover the Karpman equations. These equations are in the comoving frame of the right-going ion-acoustic wave and describe how the profile q will evolve in this frame.

The general solution of the Zakharov equations will contain both right- and left-going ion-acoustic waves. It will also contain some important interactions between these counterpropagating waves. Here, in this model, I only consider what is intuitively the dominant interaction on a unidirectional propagating ion-acoustic wave. This is done by coordinate transformation to the comoving frame of the ion-acoustic wave and averaging over a slow-time scale, ignoring the possible presence of the counterpropagating wave. However, the effects of the rf wave on the unidirectional ion-acoustic wave are accounted for. Thus, these equations are a model of only a sector of the Zakharov equations, and cannot therefore be expected to describe the solution of the full equations. However, this model does describe the evolution of a part of the solution of the Zakharov equations, namely, the evolution of an ion-acoustic disturbance into either a set of cavitons or a region of intensified oscillations.

I wish to emphasize that a large degree of incoherence and random phases does exist in this model. Neither the phase difference between the electric fields at different values of α nor the overall phase at each value of α are of consequence. Only the magnitude, which models the insensitivity of the ponderomotive force to any phase, is important.

This phase incoherence is not without its consequences. It can cause the continuous spectrum (corresponding to the linear modes) either to be exponentially damped or to undergo exponential growth with time, or both. Both can be present simultaneously (but in different regions of k space) depending on the distribution of the sources which drive this system. It is only in the case of exponential decay that one could expect the solitons to form and make their presence known as cavitons. In this system, a soliton is the same as a caviton, and thus cavitons are solitons.

Note the similarity of the above phenomena to that of "self-induced transparency."[14-16] There, one also has an exponential decay of the continuous spectrum and a phase incoherence arising from independent oscillators (in this case, independent atoms).

A detailed derivation of (1) and (2) shall be given elsewhere,[17] wherein the scalings of the various physical quantities shall be considered. Here, let me simply say that a basic derivation is given in Ref. 11, without the integral over α. This integral is then obtained by considering the rf field to be due to an incoherent collection of sources, or even just a single source, but with a finite bandwidth. The important point is that the cross terms in the ponderomotive term must average to zero because of some incoherent processes. Then Eq. (2) applies.

The parameter regime where (1) and (2) do apply is when the ponderomotive term is just able to overpower the usual ion-acoustic dispersion and nonlinear terms, but not so large that one is forced into the Yajima-Oikawa regime[10] where the slow-time derivative of the rf electric field would become important in Eq. (1).

One physical situation that these equations would model is the following: Consider an infinite homogeneous neutral plasma. Let there be initially imposed on the plasma density an arbitrary, but small, localized ion-acoustic wave. In the spirit of weakly nonlinear perturbation theory, I then use the linear theory to decompose the initial disturbance into a right-going and a left-going wave. I now treat these waves separately. Also let there be electromagnetic rf fields present with the sources of these rf fields possibly located both to the right and to the left of, but outside of, the region of the initial ion-acoustic wave. I assume the frequencies of these rf sources to be near or just above the plasma frequency of the initially infinite plasma, and that these several sources are incoherent. These sources need not be only external sources, but could be effective internal sources (i.e., other parametric processes in the plasma which generate rf waves, but are removed from the location of the ion-acoustic wave).

The method of solution of Eqs. (1) and (2) is by the inverse scattering transform.[18] First I shall briefly describe the Jost functions and the scattering data for Eq. (1) in my notation, which follows that of Kaup and coworkers[18,19]:

$$\partial_x^2 v(k,x,\tau)+[k^2-q(x,\tau)]v(k,x,\tau)=0. \tag{3}$$

Let $\phi(k,x,\tau)$ and $\psi(k,x,\tau)$ be two Jost functions which satisfy the Schrödinger eigenvalue problem (3) with $q(x)$ satisfying the Faddeev conditions.[20] The boundary conditions on these Jost functions are

$$\phi\to e^{-ikx} \quad \text{as } x\to -\infty, \tag{4a}$$

$$\psi\to e^{ikx} \quad \text{as } x\to +\infty. \tag{4b}$$

As $x\to+\infty$, the coefficients of the exponential terms are called the "scattering coefficients" and are found from

$$\phi\to ae^{-ikx}+be^{ikx} \quad \text{as } x\to +\infty, \tag{5a}$$

$$\psi\to ae^{ikx}-\bar{b}e^{-ikx} \quad \text{as } x\to -\infty, \tag{5b}$$

where the bar indicates that k is to be replaced by $-k$. Thus $\bar{b}(k)=b(-k)$, $\bar{\phi}(k,x)=\phi(-k,x)$, etc. The scattering coefficients satisfy $a\bar{a}-b\bar{b}=1$ and the Jost functions can be interrelated as $\phi=a\bar{\psi}+b\psi$, whose inverse is $\psi=a\bar{\phi}-\bar{b}\phi$. The reflection coefficient for a wave incident from $+\infty$ is $R_+=b/a$ and the transmission coefficient is $T=1/a$.

The bound-state eigenvalues are the zeros of a, which are the poles of T, for k in the upper-half complex k plane. Thus, $a(k=i\kappa_n)=0$ $(n=1,2,\ldots,N)$, where $\kappa_n>0$ and N is the total number of bound states, assumed to be finite.

The scattering data for the Schrödinger eigenvalue problem (3) may be chosen to be the set

$$S_+=\{R_+(k),\ k=\text{real};\ [\kappa_n,\rho_n]_{n=1}^N\}, \tag{6}$$

where ρ_n are the "normalization coefficients."[20,21] From S_+, one may reconstruct q.[21,22]

In order to determine the time evolution of the scattering data, one requires a "Lax pair." The first equation of this Lax pair of equations is the eigenvalue problem (3). The purpose of the second equation of the pair is to define the general time evolution of the general solution of (3). This may be taken to be $v=Av+B\partial_x v$, where $A=A_0(k,\tau)-\frac{1}{2}\partial_x B$ and B is to satisfy

$$\tfrac{1}{2}\partial_x^3 B-B\partial_x q-2q\partial_x B+2k^2\partial_x B=\partial_x\int_{-\infty}^{\infty}E^*E\,d\alpha. \tag{7}$$

Equation (7) is obtained by applying the integrability condition $\partial_\tau(\partial_x^2 v)=\partial_x^2(\partial_\tau v)$. The solution to (7) is

$$B=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\alpha}{k^2-\alpha^2}E^*E(\alpha,x,\tau), \tag{8}$$

which may be verified by substitution and use of (1).

I shall briefly outline how one determines the time dependence of the scattering data. The integral in (8) is singular, and so it is necessary to choose a contour. It is best to choose it by replacing k by $k+i0^+$. Then one must define the asymptotics of E for $x\to\pm\infty$. For a source of a scaled frequency of α at $+\infty$, the solution would be $E_+=S_+(\alpha)\phi(\alpha,x,\tau)/a(\alpha,\tau)$, where $S_+(\alpha)$ is the strength of the source. Similarly, for a source at $-\infty$, we have $E_-=S_-(\alpha)\psi(\alpha,x,\tau)/a(\alpha,\tau)$. If we take these two sources to be phase incoherent, then (8) becomes

$$B=\frac{1}{4}\int_{-\infty}^{\infty}\frac{d\alpha}{(k+i0^+)^2-\alpha^2}[\mu_+\bar{\phi}\phi+\mu_-\bar{\psi}\psi]\frac{1}{\bar{a}a}, \tag{9}$$

where $\mu_\pm(\alpha)$ are the densities of the sources and I have used $\psi^*=\bar{\psi}$, $\phi^*=\bar{\phi}$ which is valid for real q and when α is also real, as in the above integral.

Now one can determine the constant A_0 which depends on our choice of v. Letting v be ϕ and requiring $\partial_\tau \phi \to 0$ as $x \to -\infty$ gives

$$A_0 = \frac{i}{4} k \int_{-\infty}^{\infty} \frac{d\alpha}{(k+i0^+)^2 - \alpha^2} \left[\frac{\mu_+}{\bar{a}a} + \mu_- \left(1 + \frac{\bar{b}b}{\bar{a}a} \right) \right]. \tag{10}$$

As $x \to +\infty$, ϕ goes over into a linear combination of a and b, allowing us to obtain the time dependence of a and b. Thus,

$$\partial_\tau a = \frac{i}{2} k a \int_{-\infty}^{\infty} \frac{d\alpha}{(k+i0^+)^2 - \alpha^2} \left[\frac{\bar{b}b}{\bar{a}a} (\mu_- - \mu_+) \right] (\alpha, \tau), \tag{11}$$

$$\partial_\tau b = \frac{i}{2} k b \int_{-\infty}^{\infty} \frac{d\alpha}{(k+i0^+)^2 - \alpha^2} (\mu_+ + \mu_-)(\alpha,\tau) - \pi b \mu_+(k,\tau), \tag{12}$$

when k is real. When k is in the upper-half complex k plane, and off the real axis, then the last term in (12) is absent.

From the above, we may determine the time dependence of the reflection coefficient, R_+. One obtains

$$R_+(k,\tau) = R_+(k,0) \frac{e^{-(1/2)g(k)\tau}}{M(k,\tau)} e^{i[\omega\tau - \theta_a(k,\tau)]}, \tag{13}$$

where for k real

$$M(k,\tau) = [1+\Gamma(k,\tau)]^{1/2} / [1+\Gamma(k,0)]^{1/2}, \tag{14}$$

$$\Gamma(k,\tau) = \Gamma(k,0) e^{-g(k)\tau} = \bar{b}b, \tag{15}$$

$$g(k) = \pi[\mu_+(k) - \mu_-(k)], \tag{16}$$

$$\omega(k) = \tfrac{1}{2} k \mathrm{P} \int_{-\infty}^{\infty} \frac{d\alpha}{k^2 - \alpha^2} [\mu_+(\alpha) + \mu_-(\alpha)], \tag{17}$$

$$\theta_a(k,\tau) = \frac{k}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{d\alpha}{k^2 - \alpha^2} \ln M(\alpha,\tau), \tag{18}$$

and P indicates the Cauchy principal-value integral. Note the exponential time dependence in (13). When the sources on the right- and on the left-hand sides are unbalanced, $g(k)$ is nonzero, and the magnitude of $b(k)$ will change in time. An excess of sources on the left causes growth in $b(k)$ while an excess on the right causes decay. The determining factor is whether the rf wave is flowing with or against the ion-acoustic wave. We are in the frame of the ion-acoustic wave which is flowing to the right. If the rf source is on the left, then it is behind the ion-acoustic wave and the rf wave is flowing with the ion-acoustic wave. In this case, $b(k)$ grows and energy is being pumped into the ion-acoustic wave. But, if the rf source is on the right, then it is in front of the ion-acoustic wave and the rf wave is flowing against the ion-acoustic wave. Now the ion-acoustic wave is losing energy and $b(k)$ decreases.

For the bound states, we first observe that by (11), at a zero of $a(k,\tau)$ for k in the upper-half k plane, $\partial_\tau a = 0$. Thus, zeros of a do not move and are stationary. Consequently,

$$\partial_\tau \kappa_n = 0 \quad (n = 1,2,\ldots,N). \tag{19}$$

For the normalization coefficients, ρ_n, we delete the last term in (12), and obtain

$$\rho_n(\tau) = \rho_n(0) e^{\phi_n(\tau)}, \tag{20}$$

where

$$\phi_n(\tau) = \kappa_n \int_{-\infty}^{\infty} \frac{d\alpha}{\alpha^2 + \kappa_n^2} \left\{ \tfrac{1}{2} \tau [\mu_+(\alpha) + \mu_-(\alpha)] - \frac{1}{\pi} \ln M(\alpha,\tau) \right\}, \tag{21}$$

for $n = 1,2,3,\ldots,N$.

The general evolution for the solution of (1) and (2) can be quite complicated, consisting of an arbitrary assortment of growths and decays in various regions of k space, depending on the distribution of the sources. Here I shall discuss a couple of simple cases.

First, I note the inverse scattering equations for this eigenvalue problem.[21,22] Define

$$F(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, R_+(k,\tau) e^{ikx} + \sum_n \rho_n(\tau) e^{-\kappa_n x}; \tag{22}$$

then solve the following linear integral equation for $K(x,y)$:

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,s) F(s+y)\, ds = 0; \tag{23}$$

then the potential, q, is recovered from

$$q = -2\, dK(x,x)/dx. \tag{24}$$

The typical solutions for q consist of two characteristic parts: radiation (the continuous spectrum) and solitons (the bound-state spectrum).[18,22] In general, the radiation behavior is similar to the linear solution while the solitons are distinctly different and are nonlinear entities.[18,19,22] First, let us consider the case where $\mu_+(k)$

$> \mu_-(k)$ which causes an absolute decay in the continuous spectrum. This is analogous to the attenuator case of self-induced transparency.[14–16] For large τ, $R_+(k)$ will decay as $e^{-g(k)\tau/2}$, leaving only the soliton part of the solution. This solution will, in general, consist of N individual solitons, and if one waits long enough, they will separate out according to their velocities. A typical single soliton solution is

$$q = -2\kappa_n^2/\cosh^2[\kappa_n(x - x_0 - v_n\tau)], \qquad (25)$$

where x_0 is some constant and

$$v_n = \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\alpha}{\alpha^2+\kappa_n^2}[\mu_+(\alpha)+\mu_-(\alpha)]. \qquad (26)$$

Note that unlike the Korteweg–de Vries equation,[22] the largest solitons move the slowest. Each one of these solitons is a density depression which contains within itself an intensification of the electric field, E, which vanishes exponentially away from the soliton, or caviton.

In the other canonical case, when $\mu_-(k) > \mu_+(k)$, we have absolute growth in the reflection coefficient. This case corresponds to the amplifier case of self-induced transparency.[14,15,23] Now the radiation dominates the solution and after a sufficiently long time, $R_+(k) \to 1$ provided that at that value of k, $g(k) \neq 0$. And note that the rate at which $R_+ \to 1$ is governed by the value of $g(k)$ and may well differ considerably for different values of k. Nevertheless, in this case, we eventually do obtain perfect reflection after a sufficiently long time, which means that the rf wave has piled up in front of it a sufficiently high barrier of ion-acoustic plasma waves that it no longer can propagate into the plasma, and must now be totally reflected.

The above considerations only apply for a right-going acoustic wave. For left-going acoustic waves, the result is the opposite. In a real plasma, one has right-going as well as left-going acoustic waves. And in the absence of any deliberate source of rf waves (such as a microwave source at one end), the density of rf sources (such as scatterings, etc.) could be expected to be uniform throughout the plasma, or perhaps just slightly toward $\mu_+ > \mu_-$ due to Doppler shifts. Thus, one would not necessarily expect to see these dramatic growths or decays inside a typical plasma because of the left-right symmetry. However, if one places a microwave source at an edge of the plasma, he has then set up the conditions for growth in the ion-acoustic waves moving away from the source.[24] And that is what is observed.[25] The ponderomotive effect pushes the plasma away from the microwave source and the resulting structure does have the oscillatory solution characteristic of the continuous spectrum of the Schrödinger eigenvalue problem.[26]

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[1]G. J. Morales, Y. C. Lee, and R. B. White, Phys. Rev. Lett. **32**, 457 (1974).

[2]E. J. Valeo and W. L. Kruer, Phys. Rev. Lett. **33**, 750 (1974).

[3]H. C. Kim, R. L. Stenzel, and A. Y. Wong, Phys. Rev. Lett. **33**, 886 (1974).

[4]G. J. Morales and Y. C. Lee, Phys. Fluids **19**, 690 (1984).

[5]E. W. Laedke and K. H. Spatschek, Phys. Rev. Lett. **45**, 993 (1980).

[6]M. V. Goldman, Rev. Mod. Phys. **56**, 709 (1984).

[7]G. D. Doolen, D. F. DuBois, and Harvey A. Rose, Phys. Rev. Lett. **54**, 804 (1985).

[8]David Russel, D. F. DuBois, and Harvey A. Rose, Phys. Rev. Lett. **56**, 838 (1986).

[9]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].

[10]N. Yajima and M. Oikawa, Prog. Theor. Phys. **56**, 1719 (1976).

[11]V. I. Karpman, Phys. Scr. **11**, 263 (1975).

[12]E. W. Laedke and K. H. Spatschek, Phys. Fluids **23**, 44 (1980).

[13]H. Kramer, E. W. Laedke, and K. H. Spatschek, Phys. Rev. Lett. **52**, 1226 (1984).

[14]G. L. Lamb, Jr., Rev. Mod. Phys. **43**, 99 (1971), and Phys. Rev. Lett. **31**, 196 (1973), and Phys. Rev. A **9**, 422 (1974).

[15]M. J. Ablowitz, D. J. Kaup, and A. C. Newell, J. Math. Phys. **15**, 1852 (1974).

[16]D. J. Kaup, Phys. Rev. A **16**, 704 (1977).

[17]D. J. Kaup, to be published.

[18]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).

[19]D. J. Kaup and A. C. Newell, Proc. Roy. Soc. London, Ser. A **361**, 413 (1978).

[20]L. D. Faddeev, Dokl. Akad. Nauk, SSSR **121**, 63 (1958) [Sov. Phys. Dokl. **3**, 747 (1958)].

[21]I. M. Gel'fand and B. M. Levitan, Amer. Math. Soc. Transl. (2) **1**, 253 (1951).

[22]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **27**, 1095 (1967), and Commun. Pure Appl. Math. **27**, 97 (1974).

[23]S. V. Manakov, Zh. Eksp. Teor. Fiz. **83**, 68 (1982) [Sov. Phys. JETP **56**, 37 (1982)].

[24]Although we now have a boundary present, if we move into the plasma a short distance and go into the comoving frame of the inwardly going ion-acoustic wave, we would then observe the boundary moving away from us. In this case, we are approximating the condition of an infinite plasma and observe the rf wave to be propagating in the same direction as the inwardly moving ion-acoustic wave.

[25]D. L. Eggleston, A. Y. Wong, and C. B. Darrow, Phys. Fluids **25**, 257 (1982).

[26]M. J. Ablowitz and A. C. Newell, J. Math. Phys. **14**, 1277 (1977).