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Turbulent Drag Reduction by Polymers: A Quantitative Theory

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By use of the recently proposed model of the dynamics of a macromolecule in transient extensional flows, the effective viscosity increase in turbulence due to unraveling macromolecules is calculated, and a relation between this viscosity increase and a drag reduction parameter (the slope increment) is derived. The predictions of the theory are in good agreement with experiment.

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Turbulent drag reduction by polymer additives is the most spectacular effect at the interface of fluid dynamics and high-polymer physics. The pressure loss in a flow through a pipe, for example, may be reduced 3 to 4 times when an extremely small amount of flexible high polymer is added [concentration $c \simeq 10^{-6}$ g/cm^3]. A small but noticeable effect has been registered[1] at concentrations as low as 2×10^{-8} g/cm^3. The phenomenon has important applications (e.g., in oil pipelines, fire fighting, hydrotransport of solids, etc.) and has stimulated a vast amount of research since its discovery about forty years ago. In view of the extreme complexity of the physics involved, it is not surprising that only tentative explanations of the phenomenon exist.

Lumley[2] explains the drag reduction as follows: The turbulence outside the viscous sublayer at the wall may stretch the polymer chains if the strain rates in the turbulence are sufficiently large. This will lead to a higher effective viscosity in this turbulent region, and consequently to an increase in the thickness of the viscous sublayer, thus reducing the velocity gradient at the wall. The viscosity in the viscous sublayer, however, remains at the low level of the shear viscosity of the solution (practically equal to the viscosity of the solvent) because the polymer chains are not extended in the simple shear flow of the viscous sublayer, where they rotate with vorticity and thus experience the stretching and compressing effects of the strain-rate field in an alternating fashion. Thus the reduced velocity gradient at the wall means reduced shear stress at the wall τ_w, and so the reduced drag.

Lumley's hypothesis has not become a predictive theory for two reasons: (a) no prescription exists for the calculation of the viscosity increase in the turbulence, and (b) even if this viscosity increase were known, no reliable way exists to predict the drag reduction. Moreover, even the basic idea about the importance of the viscosity increase has been open to doubt, since the calculations predicting large values of the effective viscosity,[3,4] quoted by Lumley,[2] are relevant only to the nearly fully stretched polymer chains (see below), while in a turbulent flow the chains are in the state of incessant *transient* stretching and contraction, thought to occur "affinely," i.e., the elongation of the chain essentially follows the elongation of the fluid element.[5,6] This inconsistency in Lumley's picture was noticed recently by de Gennes,[6] who concluded that "the Lumley scheme (may) hold for (rigid) rods" since "they cannot follow the local deformation affinely, as done by the (polymer) coils"; for the latter de Gennes[6] proposed a completely different approach, where partially stretched chains behave elastically, and the viscosity increase is of no importance. This approach too has not led to any quantitative predictions that could be compared with experimental data. Besides, some assumptions in Ref. 6 are hardly justifiable.[7]

We will see in a moment that the basic idea of Lumley's hypothesis (though not its quantitative aspects[2]) can nevertheless be useful, provided one employs the "yo-yo" model of the polymer dynamics in transient extensional flows, proposed by Ryskin.[8] I then construct a quantitative theory of drag reduction, using this idea, using the yo-yo model to predict the viscosity increase in turbulence, and finally relating this viscosity increase to a drag reduction parameter (the slope increment).

In brief, the yo-yo model says that while the elongation of a polymer chain in an extensional flow does follow the elongation of the fluid element if the strain rate exceeds a critical value, the polymer chain does *not* deform affinely with the fluid, but *unravels*, whereby the central portion of the chain is straightened out first and then remains taut and grows in length at the expense of the two coiled portions at the ends; these end portions move apart under the influence of the flow, simultaneously diminishing in size. If the flow later becomes weak, the chain will curl back into a coil. The taut central portion generates large additional stress via a dissipative mechanism.

The yo-yo model is thus radically different from any other model of polymer dynamics, in that it predicts a very large polymer effect *during* the transient deformation of a macromolecule by the extensional flow, i.e., when the stretching is only partial. Recall that the inverse-Langevin spring of Peterlin[3] begins to produce large stress only when the macromolecule is stretched to about 90% of its contour length, while the rigid-rod result of Batchelor[4] can be applied only to "fully" stretched chains[6] (unless one invokes the yo-yo model[8]).

The fundamental reasons for this radical difference (the breakdown of the thermodynamic approach) are discussed in Ref. 8.

It is shown in Ref. 8 that in a uniaxial extensional flow the effect of polymer is equivalent to a viscosity increase, and, in particular, if $1+\zeta$ denotes the ratio of the instantaneous effective viscosity in a fluid element of the polymer solution to the viscosity of the solvent, we have

$$\zeta \simeq 0.3c[\eta]\xi^3, \qquad (1)$$

where $[\eta]$ is the intrinsic viscosity, $\xi \equiv l/l_{\mathrm{cs}}$ is the instantaneous relative elongation of a polymer chain from the beginning of the supercritical unravelling, l is the hydrodynamically effective length of the chain, $l_{\mathrm{cs}} \simeq 4R_\eta$ is the value of this length just before the beginning of the supercritical regime, and R_η is the effective radius of the coil in a quiescent fluid. The numerical factor 0.3 in (1) is actually[8] a very weak (logarithmic) function of $c[\eta]$, but we will ignore such complications here. The combination $c[\eta]$ is essentially the relative concentration c/c^*, where c^* is the overlap threshold[9]; $c/c^* \ll 1$ in dilute solutions.

The value of ξ corresponding to the full extension, $\xi_{\max}$, depends on the polymer and on the solvent quality. The total (contour) length of the chain L can be written as $L=Na$, where N is the polymerization index, and a is the maximum (stretched) length of a repeat unit, e.g., for polyethylene oxide a corresponds to a *trans* (planar zigzag) conformation and thus[10] $a \simeq 3\times(1.5\ \text{Å})\cos 35° \simeq 3.7$ Å. We also have[8]

$$R_\eta = \left(\frac{3}{10\pi}\frac{c}{n}[\eta]\right)^{1/3} = 0.46\left(\frac{M[\eta]}{N_{\mathrm{A}}}\right)^{1/3},$$

where n is the number density of the macromolecules in solution, M is the molecular weight, and N_{A} is the Avogadro number. Then

$$\xi_{\max} \simeq L/4R_\eta \simeq 0.55\{N_{\mathrm{A}}a^3N^2/M_a[\eta]\}^{1/3},$$

where $M_a = M/N$ is the molecular weight of a repeat unit. For example,[11] $\xi_{\max} \simeq 110$ for polyethylene oxide with $M = 8\times 10^6$; $\xi_{\max}$ should scale as $M^{2/5}$ in a good solvent.

In order to apply (1) in turbulence, we need some mean value of ξ (denoted as ξ_{turb}), which must be determined by $\xi_{\max}$ and the characteristics of the turbulence. We are, of course, interested in ξ_{turb} only upon onset of drag reduction, which occurs when $\tau_1 u_*^2/\nu_s \simeq 1$, where τ_1 is the longest relaxation time of the macromolecule, $u_* \equiv (\tau_w/\rho)^{1/2}$ is the friction velocity, ρ is the density, and ν_s is the kinematic viscosity of the solvent.[12,13] The coil-stretch transition[14-16] begins when the strain rate roughly exceeds τ_1^{-1}. The mean value of the microscale (Kolmogorov) strain rate $(\epsilon/\nu_s)^{1/2}$, where ϵ is the energy dissipation rate per unit mass, is about $0.5u_*^2/\nu_s$ just outside the viscous sublayer,[17] but the relevant quantity is, probably, the component of the rate of strain along the vorticity vector, since extension in other directions is impeded by the rotation of the coil with vorticity. The mean value of this component has been estimated[18] as $0.2(\epsilon/\nu_s)^{1/2}$, and is therefore about $0.1u_*^2/\nu_s$. The *peak* values are, of course, much higher, and it is the peak values that are important because of the *hysteresis* character of the coil-stretch transition—once a chain has been extended, a much smaller (by about ξ times) strain rate is required to keep it extended (or extending) than what was required to start the coil-stretch transition in the first place.[5,15,16] (The persistence time of the strain-rate field, which plays an important role in Lumley's analysis,[2] cannot be relevant since it is much longer[19] than is necessary for a fluid element, elongating exponentially in a strain-rate field, to increase in length by $\xi_{\max}$ times.)

This hysteresis occurs because the viscous pull (which tries to stretch the chain) and the entropic restoring force (which tries to bring it back to the coiled state) depend very differently on the elongation of the chain: The former grows quadratically with the elongation,[5] while the latter grows linearly at first, up to rather large elongations, then faster and faster, and finally extremely fast as the full stretching is approached.[10] The relatively slow initial growth of the restoring force leads to the hysteresis, while its eventual very fast increase implies that the final elongation of the chain will be nearly independent of the strain-rate magnitude.[5,16] This means that, *upon onset of drag reduction,* the value of ξ_{turb} should be approximately independent of the turbulence intensity,[20] and thus mainly determined by $\xi_{\max}$. (We are considering a monodisperse polymer sample here; polydispersity will lead to more and more chains of lower and lower M being stretched as u_* increases.) It will, of course, be less than $\xi_{\max}$, for two reasons: (a) Even in a steady extensional flow with high strain rate the chain elongation can never reach $\xi_{\max}$ because of the gradual increase of coiling towards the ends; it is, however, likely to approach $\xi_{\max}$ rather closely,[5,16] so this effect should not be too important; (b) because of the inherent unsteadiness of turbulence, the strain-rate component along an extended chain should eventually decrease or change sign, and so the chain should temporarily contract [but note that the direction of a material line element containing the extended chain is "always being turned towards the direction in which the rate of extension is greatest, so that the rate of extension (along this material line) . . . is a biased sample of rates of extension with a preference for large positive values"[18]]. I now introduce a numerical factor α such that $\xi_{\mathrm{turb}} \equiv \alpha\xi_{\max}$ (or, more precisely, $\langle\xi^3\rangle \equiv \alpha^3\xi_{\max}^3$). Obviously, α is less than 1. No attempt to calculate α rigorously will be made here.

We thus obtain for the polymer effect on the viscosity in turbulence, $\nu_{\mathrm{turb}} = \nu_s(1+\zeta_{\mathrm{turb}})$, the following expression:

$$\zeta_{\mathrm{turb}} \simeq 0.05\alpha^3 N_{\mathrm{A}} a^3 N^2 c/M_a. \qquad (2)$$

This completes the first part of the problem; now we need to find the relationship between the viscosity ratio ν_{turb}/ν_s and some quantitative characteristic of drag reduction. The most suitable is the *slope increment* δ, defined as follows. The classical Prandtl's resistance law for a (Newtonian) turbulent flow in a pipe is

$$f^{-1/2}=2\log_{10}(Rf^{1/2})-0.8, \qquad (3)$$

where $f\equiv 2\Delta P d/\rho U^2=8\tau_w/\rho U^2=8u_*^2/U^2$ is the friction factor, $R\equiv Ud/\nu$ is the Reynolds number, ΔP is the pressure loss per unit of length of the pipe, d is the pipe internal diameter, and U is the bulk average velocity. On the other hand, data from the drag-reduction experiments satisfy[13,21]

$$f_p^{-1/2}=(2+\delta)\log_{10}(Rf_p^{1/2})-0.8-\delta\log_{10}(Rf^{1/2})_o, \qquad (4)$$

where the subscript p stands for "polymer solution," subscript o means "at onset," and the last term owes its origin to the definition of the onset as the point where (3) and (4) cross. (Our δ is one-half that of Virk's.[21])

We shall now relate δ to ν_{turb}/ν_s via the following simple (in retrospect!) argument. The two constants, 2 and −0.8, in Prandtl's law (3) are of different nature—the first is "purely inertial" (independent of rheology), and is completely determined by the von Kármán constant (the proportionality factor between the distance from the wall and Prandtl's mixing length), while the second depends on both the von Kármán constant and the thickness of the viscous sublayer.[22] This means that if in a polymer solution flow the wall shear stress τ_w were equal to the velocity gradient at the wall times ν_{turb} (instead of ν_s), the resulting "fictitious" friction factor f_f would satisfy a resistance law similar to (3), with the same factor 2 in front of the logarithm, but with an additive constant different from −0.8. The relation between f_f and f_p is, obviously, $f_f=(\nu_{\text{turb}}/\nu_s)f_p$. We thus obtain

$$(\nu_{\text{turb}}/\nu_s)^{-1/2}f_p^{-1/2}=2\log_{10}(Rf_p^{1/2})+\text{const},$$

where const includes all additive terms. Therefore $2+\delta=2(\nu_{\text{turb}}/\nu_s)^{1/2}$ and so

$$\delta=2(1+\zeta_{\text{turb}})^{1/2}-2. \qquad (5)$$

In one important aspect the above short derivation of Eq. (5) differs from all the previous attempts[2] to relate the viscosity increase to drag reduction parameters: It is completely rigorous.

The relations (2) and (5) are the principal results of the present work; together they comprise a complete theory of the turbulent drag reduction by polymers.

The experimental data for δ are usually presented in the form of a power law. For small drag reductions (5) yields $\delta=\zeta_{\text{turb}}$, while for large ζ_{turb} we get $\delta\simeq 2\zeta_{\text{turb}}^{1/2}$. The experimental results[1] for very small drag reductions (with use of polyacrylamide, $M_a=71$, $a\simeq 2.5$ Å) were shown[23] to satisfy $\delta=9.7\times 10^6 c$, where c is in grams per cubic centimeter. The present theory predicts (if we take[1] $M=25\times 10^6$)

$$\delta=0.05\alpha^3 N_{\text{A}} a^3 M^2 c/M_a^3=8.2\times 10^8\alpha^3 c,$$

so that exact agreement with the data would result if $\alpha\simeq 0.23$.

In the crossover regime a power-law representation of (5) would give an exponent between $\frac{1}{2}$ and 1, in general agreement with the $\delta\propto c^{2/3}$ observation by Berman and Yuen.[23] For large values of δ Virk[21] found that the following correlation describes the data for polymers with methylene, oxyethylene, and siloxane backbones:

$$\delta=3.5\times 10^{-2}(N_{\text{V}}^3 c/M)^{1/2}, \qquad (6)$$

where N_{V} is the number of "backbone chain links"; in the case of the above polymers the term "link" was used by Virk to designate a single bond, so that $N_{\text{V}}=bN$, where b is the number of skeletal bonds in one repeat unit. Comparison with the present theory is facilitated if we note that for these polymers the value in centimeters of $N_{\text{A}}^{1/3}a$ is approximately equal to b. Thus the present theory predicts

$$\delta\simeq 0.45\alpha^{3/2}(b^3N^3c/M)^{1/2},$$

and so $\alpha\simeq 0.18$ would lead to exact agreement in this case.

For cellulosic-backbone polymers (guar gum and hydroxyethylcellulose) the experimental values of δ were about 20 times higher[21] than what would result from Virk's correlation (6). The reason for this becomes clear now: Virk's link is about 4 to 8 times longer in these cases, and so the factor $4^{3/2}$ to $8^{3/2}$ is missing.

Overall, the agreement with experiment is surprisingly good, especially so for a theory that does not take into account the polydispersity of the polymer sample. Whether or not this agreement is partly fortuitous can only be decided by further experiments, which can also establish the value of α with greater accuracy.

Finally, knowing δ and the onset parameters (or τ_1), one can easily compute the "percent drag reduction" $(1-f_p/f_s)\times 100\%$, where f_p and f_s correspond to the same Reynolds number. If one denotes $\lambda\equiv f_p/f_s$, and uses (3) and (4), one obtains an equation for $\lambda^{-1/2}$,

$$f_s^{-1/2}\lambda^{-1/2}+(2+\delta)\log_{10}\lambda^{-1/2}=f_s^{-1/2}+\delta[0.5f_s^{-1/2}+0.4-\log_{10}(Rf^{1/2})_o],$$

easily solvable for a given f_s by Newton's method. For small ζ_{turb} it follows from this equation that the percentage drag reduction is proportional to ζ_{turb}. The coefficient of proportionality depends on the flow conditions (the constant value of 1, or 100%, predicted for this coefficient by Lumley[24] is incorrect).

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[1] D. R. Oliver and S. I. Bakhtiyarov, J. Non-Newtonian Fluid Mech. **12**, 113 (1983).

[2] J. L. Lumley, Annu. Rev. Fluid Mech. **1**, 367 (1969), and J. Polym. Sci. Macromol. Rev. **7**, 263 (1973), and Phys. Fluids **20**, S64 (1977); J. L. Lumley and I. Kubo, in *The Influence of Polymer Additives on Velocity and Temperature Fields*, Proceedings of the International Union of Theoretical and Applied Mechanics Proceedings of the Symposium, Essen, West Germany, edited by B. Gampert (Springer-Verlag, Berlin, 1985), p. 3.

[3] A. Peterlin, Pure Appl. Chem. **12**, 563 (1966).

[4] G. K. Batchelor, J. Fluid Mech. **46**, 813 (1971).

[5] E. J. Hinch, Phys. Fluids **20**, S22 (1977).

[6] P. G. de Gennes, Physica (Amsterdam) **140A**, 9 (1986).

[7] de Gennes (Ref. 6) considered polymer chains "which belong, in real space, to eddies of size r (and which do not belong to any smaller eddy)." However, the size of a polymer chain is typically smaller than the size of a smallest (Kolmogorov) eddy, and the fluid element containing the chain "belongs" simultaneously to a hierarchy of eddies of all sizes, beginning with the smallest. The strain rate of the smallest, dissipating, eddy is the highest, and therefore only this eddy is important from the point of view of the polymer-flow interaction. Also, the scaling of the chain elongation with the macroscopic distance from the exit point in a laminar sink flow is not relevant to turbulence since the time variation of the strain rate in a convecting fluid element is entirely different.

[8] G. Ryskin, J. Fluid Mech. **178**, 423 (1987).

[9] P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell Univ. Press, Ithaca, NY, 1979).

[10] P. J. Flory, *Statistical Mechanics of Chain Molecules* (Interscience, New York, 1969).

[11] D. F. James and J. M. Saringer, J. Fluid Mech. **97**, 655 (1980).

[12] N. S. Berman, Phys. Fluids **20**, S168 (1977).

[13] N. S. Berman, Annu. Rev. Fluid Mech. **10**, 47 (1978).

[14] J. Frenkel, Acta Physicochim. URSS **19**, 51 (1944).

[15] P. G. de Gennes, J. Chem. Phys. **60**, 5030 (1974).

[16] Y. Rabin, F. S. Henyey, and R. K. Pathria, in *Polymer-Flow Interaction—1985,* edited by Y. Rabin, AIP Conference Proceedings No. 137 (American Institute of Physics, New York, 1985), p. 43.

[17] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (M.I.T. Press, Cambridge, MA, 1972), pp. 159–160.

[18] G. K. Batchelor, J. Fluid Mech. **98**, 609 (1980).

[19] H. Tennekes and J. L. Lumley, Ref. 17, p. 240.

[20] Except for the concentrations above "saturation," where the viscosity increase is so high that the strain rate drops below the value necessary to keep the chain extended (see Ref. 2). This inherent limit on drag reduction is not discussed here.

[21] P. S. Virk, AIChE J. **21**, 625 (1975).

[22] L. G. Loitsyanskii, *Mechanics of Liquids and Gases* (Pergamon, Oxford, 1966); L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1959); H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979).

[23] N. S. Berman, in *The Influence of Polymer Additives on Velocity and Temperature Fields,* Proceedings of the International Union of Theoretical and Applied Mechanics Symposium, Essen, West Germany, edited by B. Gampert (Springer-Verlag, Berlin, 1985), p. 293.

[24] See p. 283 of J. L. Lumley, J. Polym. Sci. Macromol. Rev. **7**, 263 (1973).