Geometrical Interpretation of SO(7): A Critical Dynamical Symmetry

Wei-Min Zhang and Da Hsuan Feng
Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

and

Joseph N. Ginocchio
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 24 June 1987)

Potential-energy surfaces are obtained for the three dynamical-symmetry chains of SO(8) within the fermion-dynamical-symmetry model. A remarkable feature is that the SO(7) dynamical symmetry is transitional and β soft. The transitional or "critical" behavior of SO(7) is vividly demonstrated.

PACS numbers: 21.60.Fw, 21.60.Cs, 21.60.Ev, 21.60.Jz

The use of the concept of dynamical symmetry has been one of the most important developments in nuclear-structure physics in the last decade.[1] To date, however, a dynamical symmetry, i.e., a group chain, is always associated with a particular collective mode. Thus, for the nuclear system to exhibit a phase transition, i.e., going from one collective mode to the next, it must occur from one dynamical symmetry to the next. In this Letter, we shall point out that there in fact exists a dynamical symmetry in nuclear physics, which is entirely fermionic in nature, which shows all the physical characteristics of a phase transition *within* the dynamical symmetry. We shall define such a dynamical symmetry as a *critical dynamical symmetry*.

Recently, Wu *et al.* proposed a fermion-dynamical-symmetry model[2] as a realistic model to study the phenomena of high-[3] and low-spin nuclear collective states. The basic ingredient of the fermion-dynamical-symmetry model is to construct SO(8) and Sp(6) symmetries[4] via the symmetry-dictated S ($l=0$) and D ($l=2$) fermion pairs which can be uniquely associated with the normal-parity single-particle states of the physical shell model. It was further demonstrated that nuclear data in the Pd-Ru region can be considered as empirical evidence of one of the group chains of SO(8) [heretofore referred to as the SO(7) symmetry].[5] It was suggested that this symmetry has an inherent "transitional" behavior, a behavior *unknown* to the previously studied dynamical symmetries in nuclear physics. From the presented results,[5] the predicted behavior of nuclei with SO(7) symmetry is vibrational-like for small valence-particle number and γ-unstable rotational-like [i.e., SO(6)] for large valence-particle number. It is the purpose of this Letter to explore the physical properties of this unique system and to demonstrate how its transitional character develops by studying the energy surface. To do so we use the *coherent-state method*, which is also equivalent to the most general Hartree-Fock-Bogoliubov method.

The coherent states of a quantum many-body system possessing dynamical symmetry G, and also the most general Hartree-Fock-Bogoliubov (HFB) states which conserve G, can be constructed from the so-called *Gilmore algorithm* which generalizes the well-known Glauber coherent states, where $G=\mathrm{H}(4)$,[6,7] to all compact Lie groups. The method has proven to be very powerful in the study of the phase transitions of any physical models with group symmetry.[8,9] The SO(8) coherent states $|\eta\rangle$ are constructed from the action of a displacement operator $T(\eta)$ acting on the core vacuum $|0\rangle$:

$$|\eta\rangle = T(\eta)|0\rangle = \exp(\eta_{00}S^{\dagger} + \textstyle\sum_{\mu}\eta_{2\mu}D_{\mu}^{\dagger} - \mathrm{H.c.})|0\rangle. \quad (1)$$

In Eq. (1), $\eta_{\lambda\mu}$ are twelve-dimensional parameters, T is the coset representative of the coset space Spin(8)/U(4),[10] and $S^{\dagger}$ ($D_{\mu}^{\dagger}$) creates a monopole (quadrupole) pair of fermions. Traditionally, the study of the phase transition of the nuclear system is equivalent to the study of the critical behavior of the energy surface. Here, the energy surface is the expectation value of the Hamiltonian evaluated with the coherent states: $\langle\eta|H|\eta\rangle$, where H is the SO(8) Hamiltonian[2,4]

$$H = G_0 S^{\dagger}S + b_2 P^2\cdot P^2 + b_3 P^3\cdot P^3 + b_1 P^1\cdot P^1. \quad (2)$$

In Eq. (2), P^r is the rth multipole operator of SO(8). In order for the SO(8) Hamiltonian in Eq. (2) to exhibit SU(2)⊗SO(5) (the pairing limit), SO(7), and SO(6) (the γ-unstable rotational-like limit) dynamical symmetries, the parameters (G_0, b_2) are $b_2=0$, $G_0=b_2$, and $G_0=0$, respectively. The choice of b_1 and b_3 does not affect these dynamical symmetries. Without any loss of generality, we shall take $b_1=b_3=b$. Obviously this energy surface is a function of the twelve-dimensional parameters $\eta_{\lambda\mu}$. However, the parameters $\eta_{\lambda\mu}$ can be connected more closely with the familiar "deformation" parameters of nuclei, β and γ, which we will now define.

As it turns out, one can rather easily transform $\langle\eta|H|\eta\rangle$ in terms of β and γ as follows: (a) Use the fact that the coherent state of Eq. (1), which is a HFB

state[11] for the SO(8) irreducible representation (0,0,0, $\Omega/2$) [$\Omega=\sum(2j+1)/2$, where j is the usual shell-model single-particle angular momentum[2]], is the most general intrinsic ground-state wave function within the model which must have time-reversal invariance for an even-even nucleus.[12] (b) Fix the expectation value of the valence-particle-number operator to be n. These two conditions, which will reduce the number of parameters from twelve to five, are the standard conditions imposed on the HFB theory. (c) Of the remaining five parameters, three can be considered as the Euler angles (ω) while the other two would correspond to the deformation parameters β and γ. Transforming $\langle\eta|H_n|\eta\rangle$ into the intrinsic coordinate system, we can show that for H the energy surface depends only on β,

$$E(\beta)=R(\omega)\langle\eta|H|\eta\rangle R(\omega)^{-1}, \tag{3}$$

where $R(\omega)$ is the rotational operator to transform the matrix into the intrinsic coordinate system. The deformation parameters are as follows:

$$Q_0=R(\omega)\langle\eta|P_0^2|\eta\rangle R(\omega)^{-1}$$
$$=2\Omega(n/2\Omega-\beta^2)^{1/2}\beta\cos\gamma, \tag{4a}$$
$$Q_2=R(\omega)\langle\eta|P_2^2|\eta\rangle R(\omega)^{-1}$$
$$=\sqrt{2}\Omega(n/2\Omega-\beta^2)^{1/2}\beta\sin\gamma, \tag{4b}$$

where Q_0 and Q_2 are the intrinsic quadrupole moments. In Eqs. (4), n corresponds to the nucleon number. It should be noted that the existence of the square-root factor in 4(a) and 4(b) reflects the fact that the system is made up of a finite number of fermions and hence the Pauli principle is entirely taken into account.

We can now study the energy-surface behavior of H [for SO(7)] of Eq. (2):

$$E(\beta)=A_4\beta^4+A_2\beta^2+A_0+A_\Delta, \tag{5}$$

where the parameters A_4, A_2, A_0, and A_Δ are

$$A_4=-2\Omega[G_0(\Omega+4)+4b], \tag{6a}$$
$$A_2=n[G_0(\Omega+4)+4b], \tag{6b}$$
$$A_0=G_0\left(\Omega-\frac{n}{2}-\frac{2n}{\Omega}+6\right)\frac{n}{4}+bn\left(1-\frac{n}{2\Omega}\right), \tag{6c}$$
$$A_\Delta=[G_0(\Omega+4)-4b]\frac{\Omega}{2}\left(\frac{n}{2\Omega}-2\beta^2\right)$$
$$\times\left[\left(1-\frac{n}{2\Omega}\right)^2-4\left(\frac{n}{2\Omega}-\beta^2\right)\beta^2\right]^{1/2}. \tag{6d}$$

The formulas presented in Eqs. (5) and (6) are for the SO(7) symmetry. In an analogous way, one can compute the energy surfaces for the SU(2)⊗SO(5) and SO(6) group chains as well. The details can be found in a forthcoming paper.[13]

First we shall study the transitional behavior for $b=0$. In Fig. 1, we plot the energy surfaces as functions of β for the (a) SU(2)⊗SO(5), (b) SO(7), and (c) SO(6) symmetries. For (a), which is the usual vibrational limit, the minimum energy occurs at $\beta_0=0$. For the SO(6) symmetry (c), the minimum energy exists for $\beta_0\neq0$ [$=(n/4\Omega)^{1/2}$]. An important feature of (a) and (c) is that as the particle number n increases, there is a deepening of the minimum energy, i.e., the rigidity in β_0 is enhanced. Comparing (a) and (c) to (b), the SO(7)

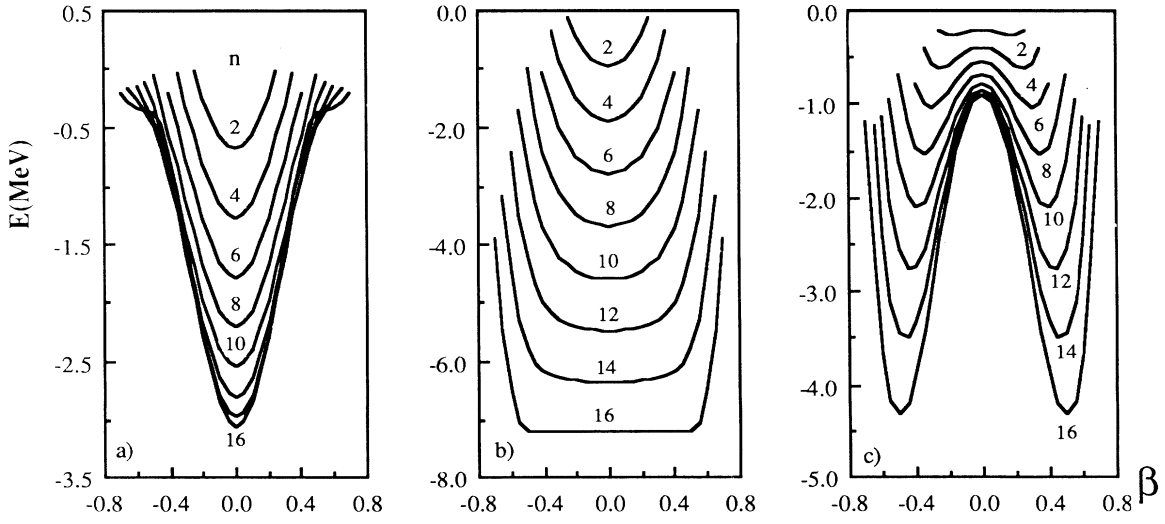

FIG. 1. Energy surfaces of the symmetries of (a) SU(2)⊗SO(5), (b) SO(7), and (c) SO(6), with $b=0$. The parameters (G_0,b_2) (in kiloelectronvolts), taken from Ref. 5, are (−45,0), (−45,−45), and (0,−45) for (a), (b), and (c), respectively. The value of $\Omega=16$. The value of n ranges from 2 to 16 in all three calculations.

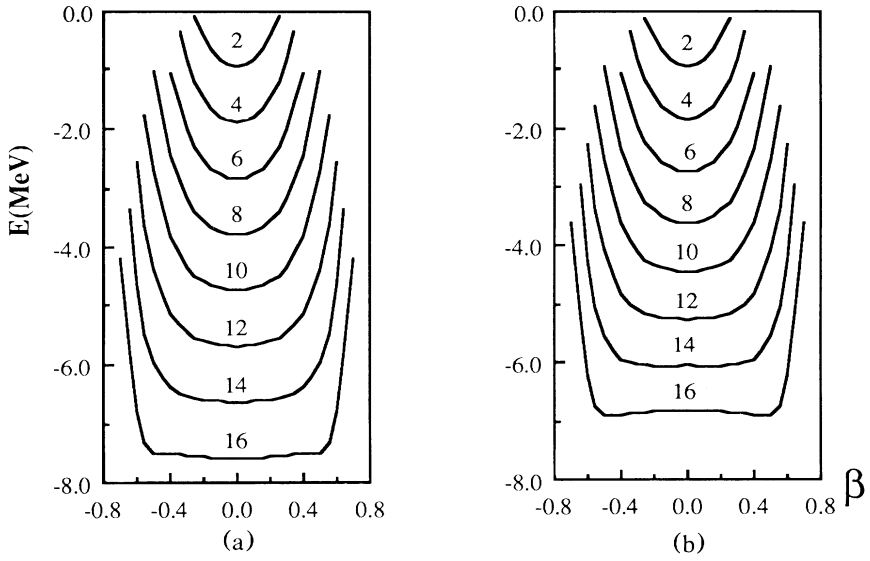

FIG. 2. Energy surfaces of the SO(7) symmetry with (a) $b > 0$ and (b) $b < 0$. The magnitude of b is 5.3 keV. All other parameters (Ω, G_0, b_2) are the same as Fig. 1(b). The value of n ranges from 2 to 16 in all three calculations.

symmetry which has $\beta_0 = 0$ for the minimum energy, we immediately see a striking difference; namely, as n increases, the system's $\beta_0 = 0$ rigidity is, in fact, relaxed for the SO(7) symmetry. Indeed, for $n = \Omega$, the energy surface of the SO(7) symmetry becomes completely "β soft" between $\beta = \pm \frac{1}{2}$.

Next we shall study the effect of the dipole and octupole interactions on the SO(7) symmetry. Indeed, as we shall now show, the β softness of the SO(7) symmetry is the reason for the critical behavior of the dynamical symmetry of Eq. (2). To show this, we shall study the SO(7) symmetry with a small value of $b \neq 0$. In Figs. 2(a) and 2(b), the results of the energy surface for the SO(7) symmetry with $b > 0$ and < 0, respectively, are presented. It is seen that for $b > 0$, the effect of the dipole and octupole terms is, in effect, to stabilize slightly further the $\beta_0 = 0$ minimum for the energy. This is especially clear when $n = \Omega$ where there is now a minimum at $\beta_0 = 0$. On the other hand, for $b < 0$, the system behaves in a rather remarkable manner; namely, for low n, it is vibrational (i.e., $\beta_0 = 0$) and for some finite value of n, which we shall denote as n_c (namely, that value of n which the second derivative of the energy surface at $\beta_0 = 0$ is zero), the system actually jumps from the vibrational behavior ($\beta_0 = 0$) to a behavior similar to the γ-unstable rotor ($\beta_0 \neq 0$). The reason why this occurs is because unlike the SU(2)⊗SO(5) symmetry [Fig. 1(a)], SO(7) symmetry [Fig. 1(b)] loses its $\beta_0 = 0$ rigidity with an increase in particle number n. Therefore, as the system becomes more β soft, any additional interaction

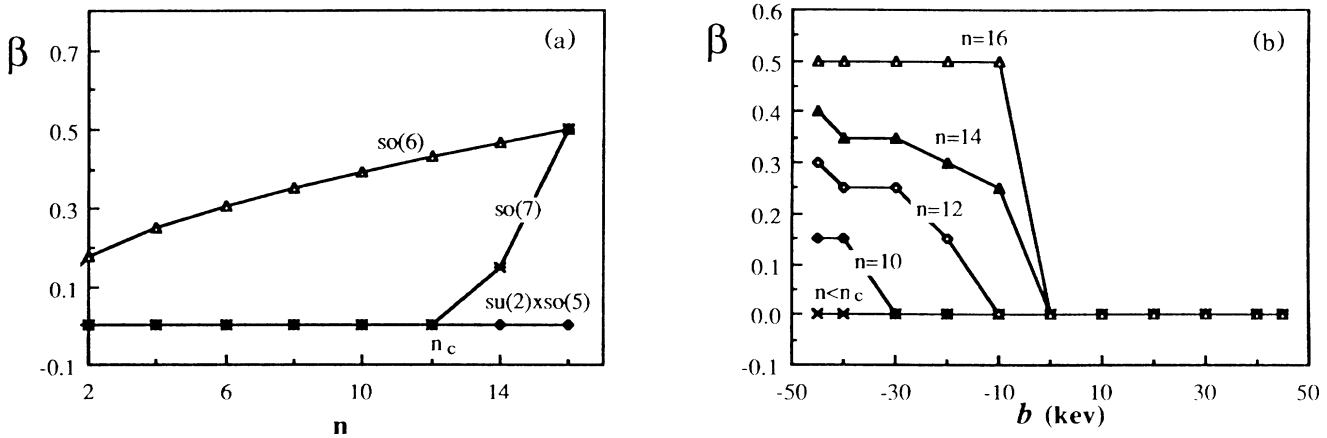

FIG. 3. (a) β value of the minimum energy plotted as a function of the particle number n, with $b = -5.3$ keV. (b) β value of the minimum energy plotted as a function of b.

which still keeps the SO(7) symmetry and which has the effect of having the $\beta_0=0$ as the *maximal point* in energy, and in this case it corresponds to $b<0$, will drive the system at some n_c away from the vibrational behavior.

One can show analytically that for the SO(7) symmetry, n_c is

$$n_c=\Omega\{1-[4b/(\Omega+4)G_0]^{1/2}\}. \quad (7)$$

Clearly, since the pairing strength G_0 is attractive, the strength b must be negative in order for a real value of n_c to exist. Therefore, the crucial strength parameter which determines the existence of the critical behavior is b. In Figs. 3(a) and 3(b), plots of the variation of β as a function of b and of n, respectively, are given. In Fig. 3(a), the $SU(2)\otimes SO(5)$ curve coincides with the SO(7) curve from zero to n_c. Beyond n_c, the SO(7) curve quickly approaches the SO(6) symmetry at $n=\Omega$. Figure 3(a) clearly demonstrates the transitional nature of the SO(7) symmetry. On the other hand, Fig. 3(b) shows that for fixed n, nonzero β_0 values for the minimum energy occur at negative values of b only, which is indeed in agreement with the finding previously shown. The energy surface of Eq. (5), which is only a function of β^2, corresponds to a second-order phase transition.[14]

In summary, the SO(7) symmetry was previously shown to be a good description of the Pd-Ru isotopes and was suggested to behave in a transitional manner. By using the coherent-states method, we have demonstrated in this Letter that it is indeed an example of the critical dynamical symmetry. This reveals (a) the possible existence of the critical dynamical symmetry in physics and (b) that the SO(7) symmetry has the characteristic of being β soft.

Discussions with R. F. Casten, Cheng-Li Wu, and Hua Wu are gratefully acknowledged. This work is supported by the National Science Foundation and the Department of Energy.

[1]*Dynamical Groups and Spectrum Generating Algebras*, edited by A. Barut, A. Bohm, and Y. Ne'eman (World Scientific, Singapore, 1987), and references therein.

[2]C.-L. Wu, D. H. Feng, X.-G. Chen, J.-Q. Chen, and M. W. Guidry, Phys. Lett. **168B**, 313 (1986), and Phys. Rev. C **36**, 1157 (1987).

[3]M. W. Guidry, C.-L. Wu, D. H. Feng, J. N. Ginocchio, X.-G. Chen, and J.-Q. Chen, Phys. Lett. **176B**, 1 (1986).

[4]J. N. Ginocchio, Ann. Phys. (N.Y.) **126**, 234 (1980).

[5]R. F. Casten, C.-L. Wu, D. H. Feng, J. N. Ginocchio, and X.-L. Han, Phys. Rev. Lett. **56**, 2578 (1986).

[6]R. Glauber, Phys. Rev. **130**, 2529 (1963).

[7]R. Gilmore, Ann. Phys. (N.Y.) **74**, 391 (1972); R. Gilmore and D. H. Feng, Prog. Part. Nucl. Phys. **9**, 479 (1983).

[8]L. M. Narducci, C. M. Bowden, V. Bluemel, G. P. Garrazana, and R. A. Tuft, Phys. Rev. A **11**, 973 (1975).

[9]J. N. Ginocchio and M. W. Kirson, Phys. Rev. Lett. **44**, 1744 (1980); A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980); D. H. Feng, R. Gilmore, and S. R. Deans, Phys. Rev. C **23**, 1254 (1981).

[10]The corresponding compact Lie group of the SO(8) algebra is Spin(8).

[11]W.-M. Zhang, D. H. Feng, C.-L. Wu, H. Wu, and J. N. Ginocchio, to be published.

[12]A. Goodman, in *Advances in Nuclear Physics,* edited by J. Negele and E. Vogt (Plenum, New York, 1979), Vol. 11.

[13]W.-M. Zhang, D. H. Feng, and J. N. Ginocchio, to be published.

[14]R. Gilmore, *Catastrophe Theory for Scientists and Engineers (Wiley, New York, 1981).*