Radiation Focusing and Guiding with Application to the Free-Electron Laser

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In the free-electron laser the interaction between the radiation and electrons can result in radiation focusing. The radiation tends to follow the electron beam when the centroid of the electron beam is laterally displaced. Spatial modulation of the electron-beam envelope induces a similar modulation in the radiation-beam envelope. These and other phenomena are studied by use of a novel source-dependent modal representation of the fully three-dimensional radiation field. Among the merits of this approach is that few modes are needed to describe the radiation accurately.

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In the one-dimensional analysis of the free-electron laser (FEL) the radiation field, wiggler field, and electron beam resonantly couple so as to modify the longitudinal wave number of the radiation field.[1-3] This resonant interaction can lead to focusing of the radiation beam. This phenomena was first analyzed for the low-gain FEL with transverse effects[4] where it was shown that the diffractive spreading of the radiation beam could be overcome by a focusing effect arising from the modified index of refraction. Experimental evidence indicating optical guiding in the FEL has also been observed.[5-7] This radiation-focusing phenomena has been shown to play a central role in the practical utilization of the FEL,[8] since, in many proposed experiments the radiation beam will not be confined or guided by a waveguide structure. Recently optical guiding in FEL's has been studied in the small-signal, exponential-growth regime,[9-12] for the asymptotic behavior of the radiation beam.

In this Letter, we present a general, self-consistent, fully nonlinear, modal representation formalism which we apply to the phenomena of radiation focusing and guiding in FEL's. The novel aspect of our modal expansion is that the characteristics of the modes are governed by the driving current density, as opposed to a heuristic numerical approach,[13] and hence it is called the "source-dependent expansion" (SDE). Instead of using the usual modal expansion consisting of *vacuum* Laguerre-Gaussian functions[14] we incorporate the source function (driving current) self-consistently into the functional dependence of (i) the radiation waist, (ii) the radiation wave-front curvature, as well as (iii) the radiation complex amplitude. Because of the source-dependent nature of our modal expansion, the fundamental mode remains dominant throughout the evolution of the radiation field. This approach, which can be applied to a wide range of problems, lends itself to fast and accurate numerical solutions as well as to a better analytical description of the FEL focusing and guiding problem. Using the SDE approach in numerical simulations of the FEL, one can efficiently incorporate simultaneously the effects of electron-beam emittance, energy spread, wiggler gradients, sideband frequencies, etc.

An envelope equation for the radiation is derived which describes the transient as well as asymptotic behavior of the radiation beam. The effects on the radiation beam of a transversely displaced electron beam as well as a longitudinally modulated electron beam have also been considered.

In our model the vector potential of the radiation field is

$$\mathbf{A}_R(r,\theta,z,t)=\tfrac{1}{2}A(r,\theta,z)\exp[i(\omega z/c-\omega t)]\hat{\mathbf{e}}_x+\text{c.c.},$$

where $A(r,\theta,z)$ is the complex amplitude, ω is the frequency, and c.c. denotes the complex conjugate. The radiation field satisfies the reduced wave equation,

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+2i\frac{\omega}{c}\frac{\partial}{\partial z}\right]a(r,\theta,z)=S(r,\theta,z),\qquad(1)$$

where $a(r,\theta,z)=|e|A/m_0c^2=|a|\exp(i\phi)$ is the normalized complex radiation-field amplitude and we have assumed that $a^{-1}\partial a/\partial z\ll\omega/c$. The source function, S, has the general form

$$S(r,\theta,z)=\left(\frac{\omega}{c}\right)^2[1-n^2(r,\theta,z,a)]a(r,\theta,z),\qquad(2)$$

where $n(r,\theta,z,a)$ is the complex index of refraction.

We choose the following representation for $a(r,\theta,z)$ in terms of associated Laguerre polynomials,

$$a(r,\theta,z)=\textstyle\sum_m\sum_p C_{m,p}(\theta,z)D_m^p(r),\qquad(3)$$

where m and $p=0,1,2,\ldots,$

$$C_{m,p}(\theta,z)=a_{m,p}(z)\cos(p\theta)+b_{m,p}(z)\sin(p\theta),\qquad(4a)$$

$$D_m^p(r)=\left(\frac{\sqrt{2}r}{r_s(z)}\right)^p L_m^p\left(\frac{2r^2}{r_s^2(z)}\right)\exp\left(-\frac{[1-i\alpha(z)]r^2}{r_s^2(z)}\right).\qquad(4b)$$

In Eqs. (4a) and (4b), $a_{m,p}(z)$ and $b_{m,p}(z)$ are complex, $r_s(z)$ is the radiation spot size, $\alpha(z)$ is related to the curvature of the wave front, and L_m^p is the associated Laguerre polynomial. The z dependence of these parameters will be determined by the source function in Eq. (1).

Substituting (3) into (1) and using the orthogonality properties of L_m^p, $\cos p\theta$, and $\sin p\theta$, we obtain

$$\left[\frac{\partial}{\partial z}+A_{m,p}(z)\right]\begin{bmatrix}a_{m,p}\\ b_{m,p}\end{bmatrix}-imB(z)\begin{bmatrix}a_{m-1,p}(z)\\ b_{m-1,p}(z)\end{bmatrix}-i(m+p+1)B^*(z)\begin{bmatrix}a_{m+1,p}(z)\\ b_{m+1,p}(z)\end{bmatrix}=-i\begin{bmatrix}F_{m,p}(z)\\ G_{m,p}(z)\end{bmatrix}, \tag{5}$$

where

$$A_{m,p}(z)=r_s'/r_s+i(2m+p+1)[(1+\alpha^2)c/\omega r_s^2-\alpha r_s'/r_s+\alpha'/2], \tag{6a}$$

$$B(z)=-[\alpha r_s'/r_s+(1-\alpha^2)c/\omega r_s^2-\alpha'/2]-i(r_s'/r_s-2\alpha c/\omega r_s^2), \tag{6b}$$

the prime denotes $\partial/\partial z$, the asterisk denotes the complex conjugate, and

$$\begin{bmatrix}F_{m,p}(z)\\ G_{m,p}(z)\end{bmatrix}=\frac{c}{2\pi\omega}\frac{m!}{(m+p)!}\int_0^{2\pi}d\theta\int_0^{\infty}d\xi\, S(\xi,\theta,z)[D_m^p(\xi)]^*\begin{bmatrix}(1+\delta_{p,0})^{-1}\cos(p\theta)\\ \sin p\theta\end{bmatrix}, \tag{6c}$$

where $\xi=2r^2/r_s^2$. The equation for $a_{m,p}$ and $b_{m,p}$ in (5) is underdetermined, since the function $B(z)$ can be shown to be arbitrary. If we choose $B(z)=0$, for example, we would in effect be expanding the radiation field in the conventional vacuum Laguerre-Gaussian modes.[14] For a source-free medium, $B=0$ would be the most appropriate choice. In the presence of a source term a more appropriate choice for $B(z)$ can be found. This is accomplished by our considering the case where the radiation beam at $z=0$ has a Gaussian radial profile symmetric about the z axis. Let us further assume that for $z>0$ the radiation-beam profile remains approximately Gaussian with a nearly circular cross section. In this case, we expect the magnitude of the coefficients, $a_{m,p}(z)$ and $b_{m,p}(z)$, to become progressively smaller as m and p take on larger values. A good approximation to the radiation beam is then given by the lowest-order mode, $a_{0,0}(z)$. From the upper component in (5), we find that only the $m=0,1$ and $p=0$ equations are relevant and they are $(\partial/\partial z+A_{0,0})a_{0,0}=-iF_{0,0}$ and $F_{1,0}=Ba_{0,0}$. The upper component in (5) provides us with a specific expression for $B(z)$ in terms of one of the moments, $F_{1,0}$, of the source term. The choice of $B(z)=F_{1,0}(z)/a_{0,0}(z)$ is source dependent and when substituted into (6b) yields first-order coupled differential equations for the parameters, r_s and α, of the Laguerre-Gaussian expansion in (3) and (4a) and (4b). Equation (5) may now be solved self-consistently for the modal coefficients $a_{m,p}$ and $b_{m,p}$.

We first consider the dynamics of an axially symmetric radiation field in the FEL. The appropriate index of refraction[4,8,10,15] for a Gaussian beam density profile is

$$n(r,z,a)=1+\frac{1}{2}\frac{\omega_b^2(r,z)}{\omega^2}\left\langle\frac{e^{-i\psi}}{\gamma}\right\rangle\frac{a_w}{|a(r,z)|}, \tag{7}$$

where

$$\omega_b^2(r,z)=\omega_{b0}^2[r_{b0}/r_b(z)]^2\exp[-r^2/r_b^2(z)],$$

$r_b(z)$ is the electron-beam radius, $r_{b0}=r_b(0)$, ω_{b0} is the initial beam plasma frequency on axis, $a_w=|e|B_w/k_w m_0 c^2$ is the normalized wiggler amplitude, γ is the electron's Lorentz factor, ψ is the electron's phase in the ponderomotive wave potential, and $\langle\,\rangle$ denotes the ensemble average over all electrons. With the assumption that in the source function the complex radiation amplitude can be approximated by the lowest-order mode, we find that (2) can be written as

$$S(\xi,z)=-4\nu\frac{a_w}{r_b^2}\frac{a_{0,0}}{|a_{0,0}|}\left\langle\frac{e^{-i\psi}}{\gamma}\right\rangle\exp\left[-\frac{1}{2}\left(\frac{r_s^2}{r_b^2}-i\alpha\right)\xi\right], \tag{8}$$

where $\nu=(\omega_{b0}r_{b0}/2c)^2=I_b/(17\times10^3)$ is Budker's constant and I_b is the electron-beam current in amperes.

An envelope equation for the radiation beam can be obtained with use of (8) and (6b),

$$r_s''+K^2(z,r_b,r_s,|a_{0,0}|)r_s=0, \tag{9}$$

where

$$K^2=(2c/\omega)^2[-1+C^2\langle\sin\psi\rangle^2+2C\langle\cos\psi\rangle+(\omega/2c)r_s^2C'\langle\sin\psi\rangle]r_s^{-4}, \tag{10}$$

and $C(z)=(2\nu/\gamma)H(z)a_w/|a_{0,0}(z)|$ measures the coupling between the radiation and electron beam, $H(z)=(1-F)/(1+F)^2$ and $F(z)=r_b^2/r_s^2$ is the filling factor. The first term on the right-hand side of (10) is the usual diffraction term, the second and third terms are always focusing, while the last term is usually a defocusing contribu-

TABLE I. FEL simulation parameters.

	Electron beam
Current	$I_b = 2$ kA ($\nu = 0.118$)
Energy	$\epsilon_b = 50$ MeV ($\gamma = 100$)
Radius	$r_{b0} = 0.3$ cm
	Radiation beam
Wavelength	$\lambda = 10.6\ \mu$m
Input power	$P(z=0) = 230$ MW [$\lvert a(0,0)\rvert = 1.84\times10^{-4}$]
Spot size	$r_s(0) = 0.6$ cm ($z_R = 10.7$ m)
	Wiggler field
Wavelength	$\lambda_w = 8$ cm
Wiggler strength	$B_w = 2.3$ kG ($a_w = 1.716$)
Resonant phase	$\psi_R = 0.358$ rad

tion. In the high-gain trapped-particle regime, $\langle \sin\psi \rangle$ and $\langle \cos\psi \rangle$ are approximately constant, while $|a_{0,0}(z)|$ increases with z. Hence, K strongly depends on z and a guided beam ($r_s' = 0$) cannot be maintained. In the low-gain trapped-particle regime $|a_{0,0}(z)|$ increases slightly and, therefore, a guided beam can be approximately achieved. In either the Compton[16] or Raman exponential-gain regime, conditions for a stable guided beam can be found.

The FEL parameters used in the following illustrations are similar to those used in Scharlemann[17] and are given in Table I where the resonant phase approximation is used and $z_R = \pi r_s^2(0)/\lambda$ is the Rayleigh length. We present first a comparison between (a) the exact numerical solution of the wave equation in (1) (using 64×64 Fourier modes), (b) the solution using a vacuum Laguerre-Gaussian modal expansion ($B = 0$, using ten modes), and (c) the solution from the Laguerre-Gaussian SDE approach ($B = F_{1,0}/a_{0,0}$, using ten modes). For an axially symmetric configuration, we show in Fig. 1 the evolution of the radiation-beam amplitude on axis obtained from methods (a), (b), and (c). The SDE solution (c) is in excellent agreement with solution (a) while solution (b), beyond a Rayleigh length, grossly deviates from (a) and (c). The excellent results obtained with the SDE approach are also reflected in the radiation-amplitude profile. Figure 2 shows the evolution of the radiation-beam radius, $1/e$ width, r_E, in the linear, exponential-gain regime of the FEL for the parameters in Table I. Five transverse modes were used in the numerical calculation.[16]

We now consider the case where the electron-beam centroid is displaced transversely in the x direction. The index of refraction in this case is given by (7) with $\omega_b^2(r,z)$ multiplied by $[1+2(r_s x_b/r_b^2)\cos\theta]$, where $x_b(z)$ is the displacement of the electron beam's centroid and $|x_b| \ll r_b$. In the FEL source term we consider only the lowest-order symmetric and antisymmetric modes with respect to the x axis. The centroid of the radiation beam is given approximately by

$$x_L(z) = \frac{r_s(z)}{\sqrt{2}} \left(\frac{a_{0,1}}{a_{0,0}} \right)_R, \tag{11}$$

where x_L is defined so that $|a|$ is proportional to $\exp\{-[(x-x_L)^2+y^2]/r_s^2\}$ and $(\ )_R$ denotes the real part. Figure 3 shows the electron- and radiation-beam centroids, $x_b = x_c[1-\operatorname{sech}(k_c z)]$ and x_L, for $x_c = r_b/4 = 0.075$ cm and $\lambda_c = 2\pi/k_c = z_R/4 = 2.7$ m. In these numerical illustrations, ten radial modes and two angular modes were used. After an initial transient, the radiation centroid is guided by and oscillates about the electron beam's centroid. We have also studied the situation where the electron-beam centroid oscillates according to

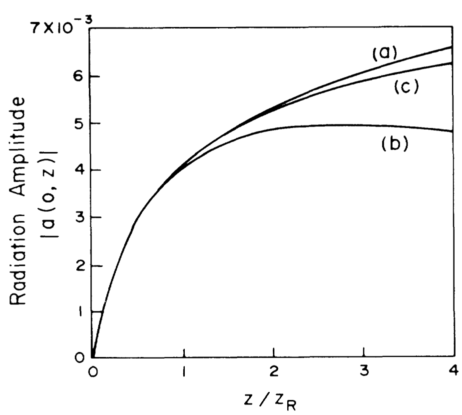

FIG. 1. Radiation amplitude on axis $|a(0,z)|$ for (a) exact numerical solution (64×64 Fourier modes), (b) vacuum modal expansion solution (ten modes), and (c) SDE solution (ten modes) at distance of $z = 4z_R = 42.8$ m.

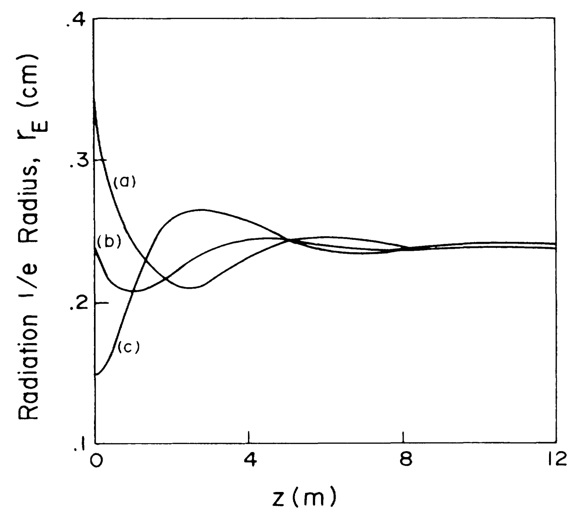

FIG. 2. Evolution of the radiation-beam radius, $1/e$ width, r_E, for initial spot sizes (a) 0.35 cm, (b) 0.24 cm, and (c) 0.15 cm.

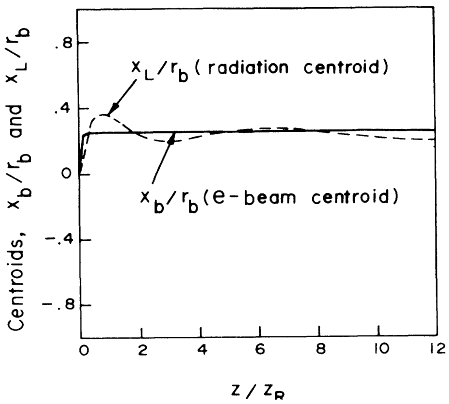

FIG. 3. Electron- and radiation-beam centroids, x_b and x_L, for a displaced electron beam, $x_b = x_c[1 - \mathrm{sech}(k_c z)]$, with $x_c = r_b/4$ and $\lambda_c = 2\pi/k_c = z_R/4$.

$x_b = x_c \sin k_c z$, with $x_c \ll r_b$ and $\lambda_c = 2\pi/k_c \lesssim z_R$. Because of the high gain in the radiation field the radiation centroid eventually follows the average position of the electron beam's centroid. When the electron-beam–centroid oscillation is due to the wiggler field, there is no change in the evolution of the radiation field.

Under certain conditions the electron-beam envelope can be spatially modulated about the z axis if the weak focusing force due to the wiggler gradient is not balanced by the defocusing forces arising from emittance and self-field effects. It can be shown that the amplitude and waist of the radiation field undergo a modulation similar to the electron-beam–envelope modulation.[16]

We have analyzed, using the SDE formalism, a number of effects associated with radiation focusing and guiding in the FEL. This approach can be readily generalized to include both spatial and temporal variations in the radiation field in order to study sideband generation and focusing effects simultaneously in the FEL.

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