

## Edge States, Transmission Matrices, and the Hall Resistance

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(Received 6 July 1987)

We consider the Landauer formula, relating conductances to transmission matrices, for a two-dimensional system in a magnetic field. We argue that the magnetoresistance,  $R$ , and the Hall resistance,  $R_H$ , satisfy the sum rule  $(R + R_H)^{-1} = (e^2/h) \text{Tr}(t^\dagger t)$  where  $t$  is the transmission matrix. For zero field our expressions reduce to the usual multichannel Landauer formulas. In the absence of dissipation,  $R$  approaches zero,  $t$  approaches a unit matrix, and quantized values are obtained for the Hall resistance.

PACS numbers: 72.10.Bg, 72.20.My, 73.20.Dx, 73.50.-h

The fact that the combination of fundamental constants,  $e^2/h$ , has the units of conductance had not been accorded much significance until recent years. An exception is the early work of Landauer,<sup>1</sup> who argued that the conductance of (spinless) electrons in a one-dimensional disordered medium is given by

$$G = (e^2/h) |t|^2 / |r|^2, \quad (1)$$

where  $t$  and  $r$  are transmission and reflection amplitudes. The Landauer formula, Eq. (1), and its generalizations to more dimensions<sup>2-6</sup> (many channels) have recently played a prominent role in the development of localization theory, both conceptually<sup>2</sup> and as a basis for numerical studies.<sup>7,8</sup> At about the same time, the quantum Hall effect, in which the Hall resistance is quantized in units of  $h/e^2$ , was discovered by von Klitzing, Dorda, and Pepper.<sup>9</sup> The appearance of the quantum unit of conductance in both cases suggests the possibility of a connection between Landauer-type formulas and the quantum Hall effect. In this Letter we present an argument which combines those customarily used to derive Landauer formulas with those used to explain the quantum Hall effect in terms of states localized near the edges of a two-dimensional system.<sup>10,11</sup>

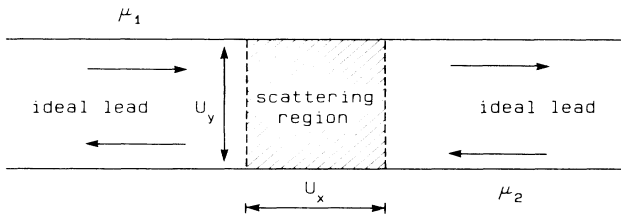


FIG. 1. In a strong magnetic field, the right-going states in the ideal leads are localized along the upper edge and the left-going states are localized along the lower edge.

We consider a two-dimensional (2D) sample of finite width in a magnetic field,  $B$ , perpendicular to the sample ( $x$ - $y$ ) plane. We take the temperature to be zero and neglect electron-electron interactions. The sample consists of an elastic-scattering region of finite length and width connected by ideal leads to particle reservoirs on the left and right (see Fig. 1). The leads are infinitely long in the  $x$  direction while, in the  $y$  direction, there is a confining potential dependent on the  $y$  coordinate only. We use a Landau gauge [ $\mathbf{A} = (-By, 0, 0)$ ] so that the eigenstates in the ideal leads are of the form

$$\psi_{nk}(\mathbf{r}_\perp) = (2\pi)^{-1/2} e^{ikx} \phi_{nk}(y), \quad (2)$$

where  $\phi_{nk}(y)$  is localized within a length on order of the magnetic length,  $l \equiv (\hbar c/eB)^{1/2}$ . The energy spectrum in the leads, illustrated schematically in Fig. 2, depends on the continuum wave-vector index and a discrete branch index, denoted by  $n$ . In the absence of the

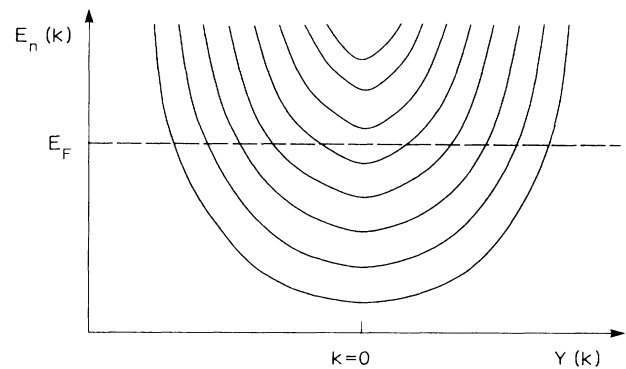


FIG. 2. Schematic energy dispersion,  $E_n(k)$ , for the ideal leads in a strong magnetic field.  $Y(k) = l^2 k$  with a gently varying confining potential. For an infinite-barrier confining potential,  $E_n(k)$  is flat unless  $Y(k)$  is within  $\approx l$  of the edge.

confining potential,  $E_n(k) = \hbar \omega_c (n + \frac{1}{2})$ , independent of  $k$ . Here  $\omega_c = eB/mc$  is the cyclotron frequency. The mean value of the velocity operator in the  $x$  direction,  $V_x$ , is given by

$$V_{nk} \equiv \langle nk | V_x | nk \rangle = \hbar^{-1} dE_n(k)/dk \quad (3)$$

and is related to the mean position in the  $y$  direction by

$$Y_{nk} \equiv \langle nk | y | nk \rangle = l^2 k - V_{nk}/\omega_c. \quad (4)$$

As  $Y_{nk}$  approaches the upper and lower edges of the sample,  $E_{nk}$  is increased by the confining potential. Thus  $V_{nk} > 0$  for the states localized along the upper edges of the ideal leads, while  $V_{nk} < 0$  for states localized along the lower edges. Each branch has positive- and negative-velocity states localized on opposite edges. It is this feature which necessitates a modification of the usual Landauer argument.<sup>12</sup>

Electrons are fed to the sample only along the upper left and lower right edges, where we take the chemical potentials to be  $\mu_1$  and  $\mu_2$ , respectively ( $\mu_1 > \mu_2$ ). An in-

coming electron in branch  $n$  on the upper left has probabilities  $T_{n'n} = |t_{n'n}|^2$  and  $R_{n'n} = |r_{n'n}|^2$  for transmission into branch (channel)  $n'$  on the upper right and reflection into branch  $n'$  on the lower left, respectively. Similarly, an incoming electron in branch  $n$  on the lower right has probability  $T'_{n'n}$  for transmission into branch  $n'$  on the lower left and probability  $R'_{n'n}$  for reflection into branch  $n'$  on the upper right. It follows that on the upper right all states are occupied up to energy  $\mu_2$ , while branch  $n'$  is occupied with probability  $\sum_n T_{n'n}$  for energies between  $\mu_2$  and  $\mu_1$ . Similarly, on the lower left, all states are occupied up to energy  $\mu_2$  while, for energies between  $\mu_2$  and  $\mu_1$ , branch  $n'$  is occupied with probability  $\sum_n R'_{n'n}$ . These results depend only on the current conservation conditions,

$$\sum_n (T_{n',n} + R'_{n',n}) = \sum_n (T'_{n',n} + R_{n',n}) = 1, \quad (5)$$

which require an outgoing channel to be fully occupied if all incoming channels at that energy are fully occupied. The current through the system is conveniently calculated in the right-hand ideal lead,<sup>13</sup>

$$I = e \sum_{n'} \int \frac{dk}{2\pi} P_{n'}(k) V_{n'k} = \frac{e}{2\pi} \sum_{n',n} T_{n',n} (\mu_1 - \mu_2) \frac{V_{n',k}}{dE_{n'}(k)/dk} = \frac{e(\mu_1 - \mu_2)}{h} \sum_{n',n} T_{n',n} = \frac{e}{h} (\mu_1 - \mu_2) \text{Tr}(t^\dagger t), \quad (6)$$

where we have used Eq. (3) to obtain the usual cancellation between the channel velocity and its density of states. In Eq. (6),  $P_{n'}(k)$  is the probability that branch  $n'$  is occupied at wave vector  $k$ . The same calculation in the left lead shows that current conservation requires

$$\sum_{n',n} (T_{n',n} + R_{n',n} - \delta_{n',n}) = 0. \quad (7)$$

We now calculate the chemical potentials which would be measured in the upper right ( $\mu_{UR}$ ) and lower left ( $\mu_{LL}$ ) channels assuming that some inelastic scattering is present which brings these channels into thermodynamic

equilibrium without altering the carrier density. Elementary calculations give

$$\mu_{LL} = \mu_2 + (\mu_1 - \mu_2) \left[ \sum_{n,n'} V_n^{-1} R_{n,n'} \right] / \sum_n V_n^{-1} \quad (8a)$$

and

$$\mu_{UR} = \mu_2 + (\mu_1 - \mu_2) \left[ \sum_{n,n'} V_n^{-1} T_{n,n'} \right] / \sum_n V_n^{-1}, \quad (8b)$$

where the wave-vector index on the velocity in the  $n$ th band has been left implicit. It follows that the Hall resistance is given by

$$R_H = V_H/I = (\mu_1 - \mu_{LL})/eI = (h/e^2) \left[ \sum_{n,n'} (\delta_{n',n} - R_{n',n}) V_n^{-1} \right] \left[ \sum_n V_n^{-1} \sum_{m',m} T_{m',m} \right]^{-1} \quad (9)$$

and the magnetoresistance by

$$R = V_x/I = (\mu_1 - \mu_{UR})/eI = (h/e^2) \left[ \sum_{n,n'} (\delta_{n',n} - T_{n',n}) V_n^{-1} \right] \left[ \sum_n V_n^{-1} \sum_{m',m} T_{m',m} \right]^{-1}, \quad (10)$$

while

$$G \equiv (R + R_H)^{-1} = (e^2/h) \text{Tr}(t^\dagger t). \quad (11)$$

This sum rule reflects the fact that the chemical potential difference between reservoirs on the left and right is  $e(V_x + V_H)$ . It follows from Eq. (10) that in the quantum Hall regime, defined by the absence of dissipation ( $R=0$ ),  $T_{n,n'} = \delta_{n,n'}$ . Combined with current conservation [Eq. (7)] this gives  $R_{n,n'} \equiv 0$  and, hence,

$$R_H = h/Ne^2, \quad (12)$$

where  $N$  is the number of branches crossing the Fermi energy in the ideal leads. On the other hand, when the magnetic field vanishes, left- and right-going states in each lead will come into equilibrium at chemical potentials

$$\mu_L = \frac{1}{2} (\mu_1 + \mu_{LL}) = \mu_2 + (\mu_1 - \mu_2) \left[ \sum_{n,n'} V_n^{-1} \times \frac{1}{2} (\delta_{n,n'} + R_{n,n'}) / \sum_n V_n^{-1} \right] \quad (13a)$$

on the left side and

$$\mu_R = \frac{1}{2}(\mu_L + \mu_{UR}) = \mu_2 + (\mu_1 - \mu_2) \left( \frac{\sum_{n,n'} V_n^{-1} \times \frac{1}{2} T_{n,n'}}{\sum_n V_n^{-1}} \right) \quad (13b)$$

on the right side. In this case the conductance is given by  $G = eI/(\mu_L - \mu_R)$  and the multichannel Landauer formula is recovered in the form advocated by Azbel<sup>5</sup> and Büttiker *et al.*<sup>6</sup>

In our argument we have supposed that the left-going and right-going states on each side of the scattering region are not in equilibrium with each other. For truly ideal leads and samples much wider than a magnetic length, this can be expected on the basis of the localization of states near the sample edges whenever  $E \neq \hbar\omega_c(n + \frac{1}{2})$ . Our argument should apply whenever the Landau levels are well defined ( $k_B T \ll \hbar\omega_c$  and  $\omega_c \tau \gg 1$ ), the leads are electronically two-dimensional, and a local equilibrium is established on each edge of the sample. It is particularly well suited to describe recent experiments on the quantum Hall effect in narrow channels.<sup>14-16</sup>

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<sup>13</sup>We assume that the transmission matrix and the branch velocity are independent of energy over the range of interest. The spin index is implicit in the ideal-lead branch index, and the wave-vector index on the velocities can be dropped.

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