Self-Induced Spatial Disorder in a Nonlinear Optical System

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An optical ring cavity containing distributed nonlinear elements is proposed as a promising candidate for investigation of the dynamic stability of spatial disorder in a system far from thermal equilibrium. If the interaction between the elements is unidirectional, the stability of disordered structure can be determined by the spatial Lyapunov exponent. This fact implies that spatial disorder is frozen under quite restricted conditions, and most of the spatially disordered structure is replaced by spatiotemporal chaos. However, in the case of a bidirectional interaction, the spatial disorder is self-induced over a wide range of the control parameter.

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It has recently been recognized that many temporally irregular phenomena consist of nothing more than the intrinsic chaos inherent in the rules of their evolutionary processes. The question then arises as to whether it is possible to understand spatially irregular structures in terms of the intrinsic chaos inherent in the rule which determines their spatial arrangements. This concept is appealing. In fact, complicated structures which are typical of the commensurate-incommensurate phase transition in equilibrium systems have been elucidated in terms of dynamical theory based on this idea.[1,2] In equilibrium systems, however, the spatial chaos derived from the rule of spatial arrangement is unlikely to be a true ground state and is more probably a thermodynamically metastable state.[1] If so, the possibility arises that spatial disorder might be more easily realized in systems far from thermal equilibrium such as when energy is injected from the outside. Moreover, in macroscopic nonequilibrium systems, the dynamical stability rather than thermal stability of spatial structures is essential. It is well known that spatial periodic structures can be stabilized dynamically in nonequilibrium systems (dissipative structures).[3] Spatiotemporal randomness (chaos) has been found to take place in such systems as the injected energy increases.[4] These results appear to provide implicit support for the existence of spatial chaos in nonequilibrium systems. On the other hand, another possibility exists in which spatial chaos is dynamically unstable and resolves itself into spatiotemporal behavior. In any case, the dynamical stability of spatial disorder remains unknown at present.

The possibility of the existence of spatial disorder (chaos) has been predicted in a nonlinear optical system, which is a typical example of nonequilibrium physical systems.[5] However, the dynamical stability of such spatial disorders, which is the essential problem, is not clearly understood. In this Letter, we introduce a simple dynamical system which can easily be realized with collective nonlinear optical elements. The distinct feature of this model is that the stationary solutions imply spatial chaotic solutions within certain limits. Using the proposed model, we investigate the following issues: (1) Is the spatial disorder stable? (2) If so, is the spatial disorder accessible by a physical manipulation? (3) If spatial disorder is not stable what phenomena take place? We will show that spatial disorder exists quite stably over a comparatively wide range of the control parameter and that it can be easily realized within certain limits.

The conceptual model is shown in Fig. 1. In the mod-

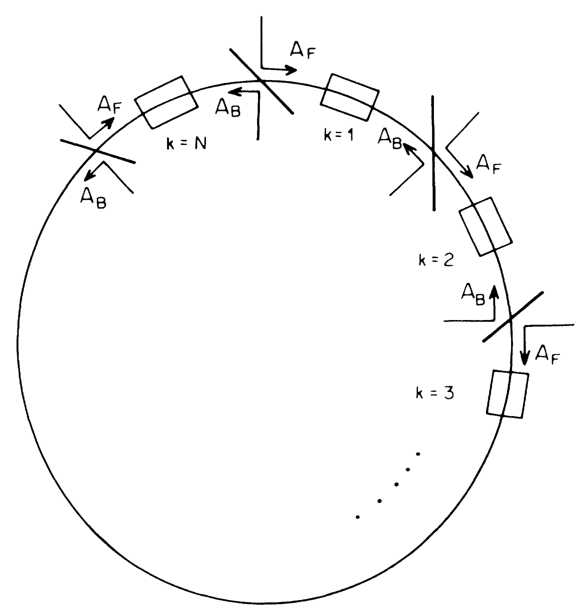

FIG. 1. Conceptual model of an optical bistable system with distributed nonlinear elements.

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el, nonlinear elements possessing a third-order susceptibility are arranged in a looped optical ring cavity. These elements interact via counterpropagating light beams which are introduced through the mirrors separating the elements. For the sake of simplicity, we will first examine the unidirectional excitation case, that is $A_F = A$ and $A_B = 0$.

The dynamics is described by the following coupled difference-differential equations[6]:

$$E_{k+1}(t) = A + BE_k(t - L/c)\exp\{i[\phi_k(t) + \phi_0]\}, \quad E_{N+1} = E_1; \tag{1}$$

$$\tau\dot{\phi}_k(t) = -\phi_k(t) + |E_k(t - L/c)|^2, \quad \phi_{N+1} = \phi_1, \quad k = 1,2,3,\ldots N. \tag{2}$$

Here, ϕ_k is the phase shift which is introduced into the field when it passes across the kth element, ϕ_0 is the linear phase shift, $B = \sqrt{T}$ is the coupling coefficient between adjacent cells (T is the mirror transmittance), L is the cell length, c is the velocity of light, and τ is the medium response time. In addition, we assume a lossless medium. Generalization to a lossy medium is straightforward.

Within the limit of large dissipation, i.e., $B \ll 1$ and $A^2B \simeq 1$, and by elimination of E_k adiabatically, Eqs. (1)-(2) can be simplified as

$$\tau\dot{\phi}_{k+1}(t) = -\phi_{k+1}(t) + f_A(\phi_k), \quad \phi_{N+1} = \phi_1,$$
$$f_A(\phi) = A^2[1 + 2B\cos(\phi + \phi_0)]. \tag{3}$$

Here, we assume a very thin medium $[\tau/(L/c) \gg 1]$ and neglect the delay time.

The stationary solution, $\bar{\phi}_k$, of Eq. (3) is determined by the simple mapping rule

$$\bar{\phi}_{k+1} = f_A(\bar{\phi}_k), \quad \bar{\phi}_{N+1} = \bar{\phi}_1. \tag{4}$$

It is apparent that these solutions are "chaotic" if the parameter $2A^2B$ is made sufficiently large. Therefore, stationary solutions of Eq. (3) are likely to exhibit spatial disorder. However, the most important point is the dynamical stability of $\{\bar{\phi}_k\}$.

First, let us carry out the linear-stability analysis. This is quite straightforward. Given a small initial deviation, the growth rate of the deviation from the stationary solution $\delta\phi \simeq \exp(\lambda t)$ can be expressed in terms of the following characteristic equation:

$$(\lambda + 1)^N = \exp(N\alpha)\,\mathrm{sgn}\,\sigma, \tag{5}$$

where

$$\sigma = \prod_{k=1}^{N} f_A'(\bar{\phi}_k), \quad \alpha = (1/N)\sum_{k=1}^{N} \ln|f_A'(\bar{\phi}_k)|.$$

Thus Eq. (5) tells us that the dynamical stability of spatial structure $\{\bar{\phi}_k\}$, which is evaluated by the sign of the real component of N λ's, is closely related to the stability of the map $\bar{\phi}_{k+1} = f_A(\bar{\phi}_k)$ through the "spatial" Lyapunov exponent α. From the above analysis, it is easy to show the following result. If the spatial structure $\bar{\phi}_k$ is the stable solution of the map $\bar{\phi}_{k+1} = f_A(\bar{\phi}_k)$, i.e., $\alpha < 0$ for $N \gg 1$, every λ has a negative real component and the spatial structure is stable.[7]

Stable solutions of the mapping rule $\bar{\phi}_{k+1} = f_A(\bar{\phi}_k)$ ($\alpha < 0$) can be classified into two classes. One implies the period-one cycle solution connected to the trivial solution of $\bar{\phi}_k = 0$ as $A \to 0$ and 1×2^n cycle solutions period doubled from the period-one cycle solution. This sequence is referred to as $S(1)$. The other class of solution consists of period-N and period-p cycle solutions (p any integer $\neq 1, N$) which appear via tangential bifurcation, and $p \times 2^n$ cycle solutions period doubled from the period-p cycle solutions. This sequence is referred to as $S(p)$. In general, period-q cycle solutions (q a divisor of N) which satisfy the boundary condition can exist as stable spatial structures. Results of the simplistic case where $N = 2^1 \times 3$ are shown in Fig. 2(a), where it is assumed that $B = 0.1$ and $\phi_0 = 0$. Spatial period doubling initially takes place as A increases and the period-two structure is frozen in this case.[8] Is the $S(p)$ structure realized as A increases up to the stable domain of $S(p)$? This is not the case and dynamical instability, which leads to spatiotemporal chaos (STC) when $N \gg 1$, develops instead. This is because $S(p)$ solutions, including the unstable regions, form closed loops (isolas) and these are isolated from other isolas as well as from $S(1)$ [see Fig. 2(a)]. In addition, their stable regions localize at the edges of isolas where the tangential bifurcation takes place. Therefore, the STC connected with $S(1)$ can coexist with $S(p)$. As p increases, the stable regions of $S(p)$ decrease exponentially. Accordingly, the basin of attraction of $S(p)$ decreases exponentially and becomes narrower than that of STC. This means that one has to set the initial conditions sufficiently close to $S(p)$ patterns in order to realize these patterns. Otherwise, the STC which connects with $S(1)$ is realized instead. However, STC is loosely related to $S(p)$. In short, when A exceeds the narrow $S(p)$ stable region, the motion of STC in phase space is attracted by the unstable branches of isolas on which $S(p)$ lies and the system tends to wander around these unstable branches as shown in Fig. 2(b). This indicates that $S(p)$ structures can be realized transiently although they cannot be frozen stably.[9]

In the unidirectional excitation discussed so far, the information flow is restricted to one direction. When one introduces counterpropagating beams A_B into the system, the dynamics is dramatically changed. The governing equation is modified as[10]

$$\tau\dot{\phi}_k^{F\text{ or }B} = -\phi_k^{F\text{ or }B} + f_A(\phi_{k-1}^F) + f_A(\phi_{k+1}^B) \tag{6}$$

under the same conditions as the unidirectional case. Here, the effect of nonlinear refractive index grating is

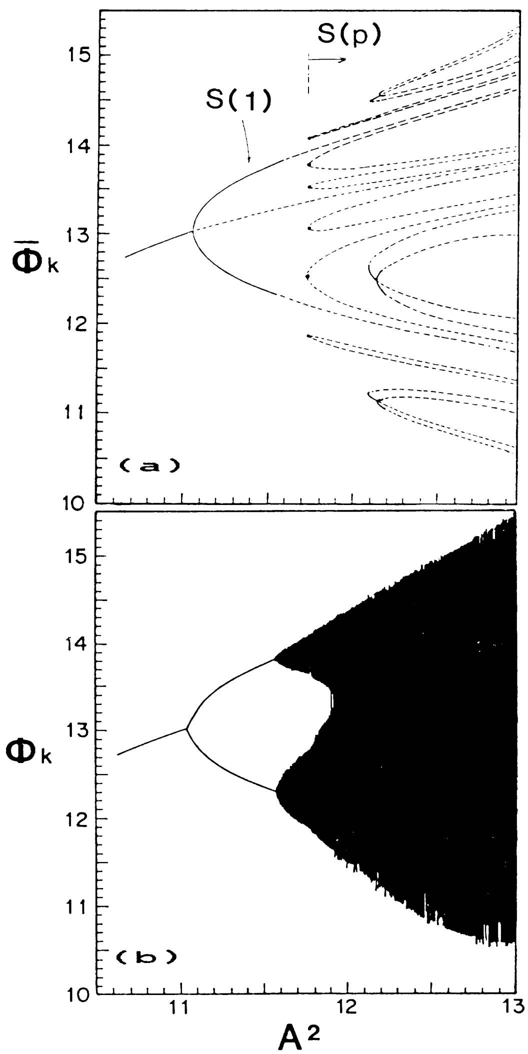

FIG. 2. (a) Spatial bifurcation diagram for $B=0.1$, $\phi_0=0$, and $N=2^1\times 3$. Solid and dotted curves correspond to stable and unstable solutions, respectively. (b) ϕ_k $(k=1,2,\ldots,N)$ vs A^2 for $N=2^1\times 3$.

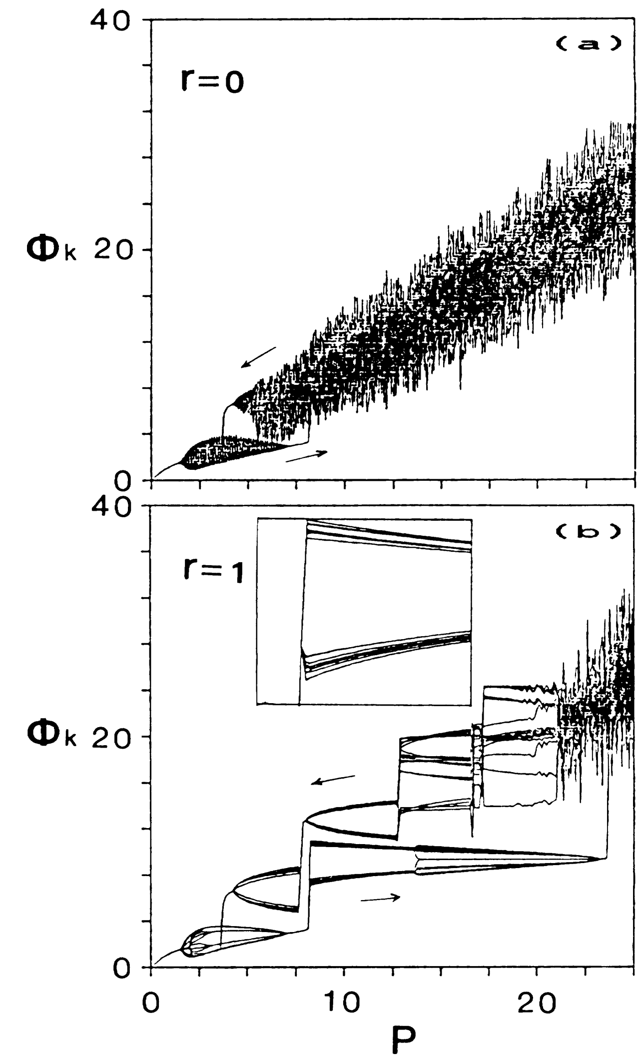

FIG. 3. ϕ_k $(k=1,2,\ldots,N)$ vs P. $B=0.3$, $\phi_0=0$, and $N=23$. (a) $r=0$ (unidirectional) and (b) $r=1$ (bidirectional). Inset: Enlargement around $P=10$. In this case, symmetric period-N solutions are realized and bifurcation into twelve different states is seen.

neglected, on the assumption of fast spatial diffusion of the excited states. From Eq. (6) $\phi_k^F=\phi_k^B(t\to\infty)$. We define ϕ_k^F and ϕ_k^B as ϕ_k hereafter. Even if $r=A_F^2/A_B^2$, which represents the symmetry of the system (≤ 1), is equal to unity, Eq. (6) lacks the potential condition of $\partial\dot\phi_k/\partial\phi_{k-1}=\partial\dot\phi_{k-1}/\partial\phi_k$ and therefore there is no Lyapunov functional corresponding to a Hamiltonian (or free energy) in equilibrium systems. Consequently, this fact does not ensure an approach to static configurations. Indeed, the STC is realized in the high-intensity regime as it is in the case of unidirectional interaction. In the low-intensity regime, however, period-N cycle spatial solutions are found to be stabilized in wide regions!

ϕ_k are plotted, except for transients, as $P=A_F^2+A_B^2$ is increased (decreased) very slowly in the case of $r=1$ [Fig. 3(b)]. ϕ_k varies stepwise with P, being accompanied by hysteresis, and on the lower-intensity side of each step, ϕ_k is bi(multi)furcated into N different static values. Moreover, the global structure does not depend upon the system size. (We examined a system size of $N<700$.) These spatial structures are totally different from those of $S(p)$ observed in the case of $r=0$. Since Eq. (6) does not have a Lyapunov functional, it is very difficult to judge rigorously whether the period-N cycle solution is really the "ground state" or not. However, it is worth noting that the period-N cycle solution is frozen and that it does not collapse into other structures, even under the presence of external noise. An example of

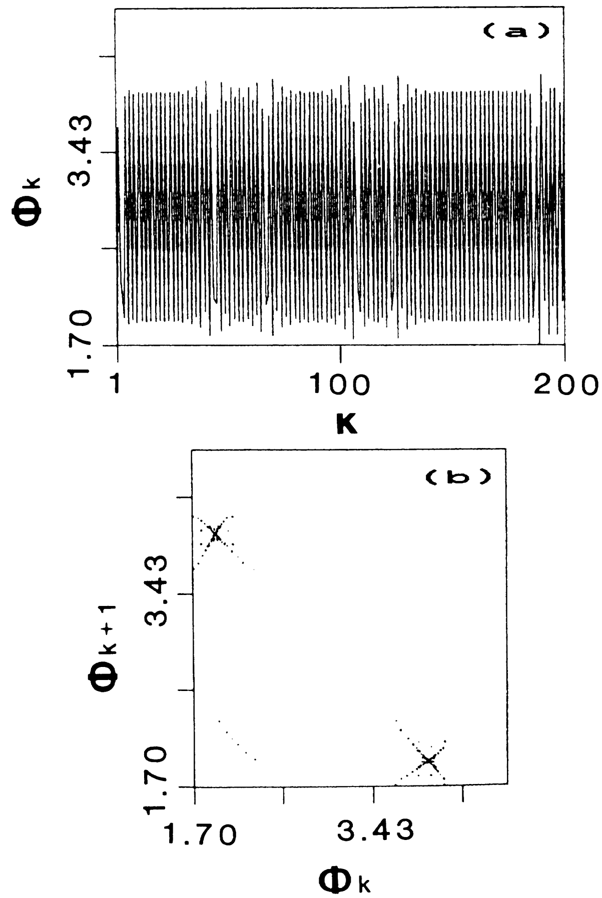

FIG. 4. An example of frozen disordered patterns which is self-induced by an increase of P to 6, where $N=200$ and $r=1$. (a) Self-induced spatial disorder, (b) spatial return map.

disordered structures and the corresponding "spatial' return map are shown in Fig. 4. The spatial displacement of ϕ_k seems to have an almost period-two cycle structure into which "holes" like intermittent bursts are inserted. The return map suggests that such holes are due to homoclinic orbits originating from a period-two fixed point. Indeed, period-two cycle solutions are proved to be absolutely stabilized as a result of the local feedback for counterpropagating fields. Moreover, the "hole" structure in Fig. 4 can also be shown to be a homoclinic orbit which is asymptotic to the period-two cycle solution.

As for large N, an extremely large variety of N-cycle spatial structures is expected to exist. Indeed, various heteroclinic structures, which connect different period-two cycle solutions, have been found to be frozen in addition to the homoclinic structures at high P regions near STC. This indicates that STC strange attractors are constructed, which are spread over the basin of attractions of coexisting N cycle structures. This might suggest that STC is interpreted as a "heteroclinic trajectory" which dynamically connects destabilized spatial chaotic structures.

In any event, the study of the relationship between STC and the large number of coexisting frozen random patterns may provide an important clue for the understanding of turbulence phenomena. This subject is now under investigation.

Finally, it should be pointed out that the proposed system possesses interesting features with respect to practical application, as well as the academic aspects discussed in this Letter. In fact, novel cooperative functions such as flip flops and assignment to spatial patterns in hysteretic regions are proved to be realizable by modulation of only a few cells in the case of unidirectional interaction.[11] A variety of coexisting patterns observed in bidirectional interaction serve as the basis for a novel optical memory device which stores complicated input information as spatial patterns.

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[8]As A increases into the period-four cycle region, which does not exist in the steady state for $N=6$, the ϕ_k of every other cell tends to wander between the upper and lower branches bifurcated from each period-two cycle solution at every round trip [see Fig. 2(b)].

[9]When the system size N is sufficiently small, $S(p)$ structures can be realized by the modulation of the input field of several elements.

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