Vortex-Ring Model of the Superfluid λ Transition

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An initial model of the superfluid λ transition is constructed with use of vortex-ring excitations, as originally suggested by Onsager and Feynman. A real-space renormalization technique generates a screened vortex energy and core size, and gives rise to a transition where rings of infinite diameter are excited as the superfluid density approaches zero at T_c. Although the model satisfies the Josephson hyperscaling relation, it is not yet a complete theory: The superfluid density exponent is $\nu=0.53$, and does not match the known value $\nu=0.67$.

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The superfluid transition of liquid ^{4}He has been studied intensively for nearly 50 years. In spite of this, there is still little known about the underlying physical mechanism. Onsager and Feynman[1] proposed in their original papers that vortex excitations might be responsible for the transition. Since then many authors have reiterated this proposal, and at least a partial listing of these papers is given in the references of Kotsubo and Williams.[2] The key role of vortex-ring excitations, in particular, was emphasized by Popov,[3] Wiegel,[4] Banks, Myerson, and Kogut,[5] and Nelson and Toner.[6] These papers gave a qualitative and physical picture of the transition, but quantitative predictions remained elusive. On the other hand, perturbation-series techniques such as the high-temperature expansion and the $4-\epsilon$ expansion proved to be successful in the calculation of the critical exponents of the transition.[7] However, these are formal and mathematical procedures that are hard to apply in many situations, and the momentum-space representation that is used makes it difficult to identify the excitations causing the fluctuations.[8]

The role of vortices in the superfluid transition has recently been made much more concrete by the numerical results of Kohring, Shrock, and Wills,[9] who used a Monte Carlo simulation of the three-dimensional (3D) *XY* model to give strong evidence that vortices play a crucial part in the transition. Vortices had previously only been known to be important in two dimensions (2D), as shown theoretically by Kosterlitz and Thouless[10] and as confirmed by experiments in thin helium films.[11] The question of how the transition evolves as one goes from 2D to 3D geometries has recently arisen in experiments on helium films adsorbed in packed powders.[2] The results there led to speculation that the nature of the transition did not seem to change much as one varied the film thickness to the point where the powder pores fill completely with helium. The finite-size 2D transition appears to merge smoothly with the finite-size 3D transition, with little sign that the vortices cease to play a role in the 3D case. This conclusion has now been corroborated by the Monte Carlo results.

In this Letter the superfluid density is calculated near the transition by consideration of the screening effects of circular vortex rings on an applied superflow. The starting point is to realize that vortex rings are eigenfunctions of the Landau-Ginzburg-Wilson Hamiltonian.[12] The "bare" energy of a circular ring of radius R is given by (in dimensionless units, normalized by k_BT)

$$U_0=2\pi^2(\hbar^2/m_4^2)(\rho_s^0/k_BT)R\{[\ln(R/a)+C]+\ldots\}, \tag{1}$$

where m_4 is the He atom mass, ρ_s^0 is the bare superfluid density, and C is a constant related to the core energy; the calculation of Roberts and Grant[12] gives $C=0.464$. The core radius a is related to the superfluid density and the ϕ^4 coupling constant V_0 by $a=(\hbar^2/2V_0\rho_s^0)^{1/2}$. The higher-order terms in the expression for U_0 are functions only of a/R, and vanish in the limit $R\rightarrow\infty$. The first correction term for the classical vortex ring is of order[13] $(a^2/R^2)\ln(R/a)$, and a similar form is thought to hold for the Landau-Ginzburg-Wilson rings,[12] although only a numerical evaluation[14] is available for that case.

A magnetostatic analogy can be exploited to regard the ring as a dipole current loop,[15,16] having a dipole moment $\mathbf{m}$ and an impulse $\mathbf{p}$. For a large ring ($R\rightarrow\infty$), $p=4\pi\rho_s^0 m$, and $m=\pi\hbar R^2/2m_4$. In a flow field $\mathbf{v}_s$, the ring energy is lowered to $U'=U_0-\mathbf{p}\cdot\mathbf{v}_s$, and following Kosterlitz and Thouless,[10] the polarizability of a large ring is

$$\alpha=\frac{1}{v_s}\frac{\int_0^{2\pi}d\theta\, m\cos\theta\sin\theta\exp(pv_s\cos\theta/k_BT)}{\int_0^{2\pi}d\theta\sin\theta\exp(pv_s\cos\theta/k_BT)}=\frac{\pi^3}{3}\frac{\hbar^2}{m_4^2}\frac{\rho_s^0}{k_BT}R^4. \tag{2}$$

For finite R the first correction to this is of order $(a^2/R^2)[\ln(R/a)]^2$.

The number of rings per unit volume thermally excited at a given point, with radius between R and $R+dR$, taking the minimum diameter to be $2a_0$ and counting the ways of orienting a loop, is

$$dn(R)=[4\pi R^2 dR/(2a_0)^6]e^{-U_0(R)}. \tag{3}$$

The length-dependent susceptibility is then $\chi=\int_{a_0}^R \alpha \times dn(R)$. An applied flow will be screened by the current resulting from the oriented dipoles, reducing the superfluid density. It is useful to express this screened density in dimensionless form by defining $K_r=(\hbar^2/m_4^2)\times(\rho_s a_0/k_B T)$ and $K=K_r\rho_s^0/\rho_s$, and the permeability is then given by

$$\mu=K/K_r=\rho_s^0/\rho_s=1+4\pi\chi, \tag{4}$$

yielding for the observable superfluid density ($R\to\infty$)

$$\frac{1}{K_r}=\frac{1}{K}+\int_{a_0}^{\infty}\frac{dR}{a_0}A\left(\frac{R}{a_0}\right)^6\exp\left[-2\pi^2K\frac{R}{a_0}\left(\ln\frac{R}{a}+c\right)\right], \tag{5}$$

with $A=\pi^5/12$, and where for the time being we neglect the correction terms in the vortex energy and polarizability. It is convenient in what follows to define two further scaling variables y and g by the relations $g=(2a_0m_4^2k_BT/\hbar^4)V_0=a_0^2/ka^2$ (where g is the dimensionless coupling constant) and $y=\exp[-2\pi^2K(\ln\sqrt{gK}+C)]$. Equation (5) can then be written as

$$\frac{1}{K_r}=\frac{1}{K}+\int_{a_0}^{\infty}\frac{dR}{a_0}A\left(\frac{R}{a_0}\right)^6y^{R/a_0}\exp\left(-2\pi^2K\frac{R}{a_0}\ln\frac{R}{a_0}\right). \tag{6}$$

To make this expression self-consistent, it is necessary to replace K in the vortex energy of the exponential with the screened value K_r.[17] At low temperatures the integral is small, and the exponential can be expanded in powers of the integral. Equation (6) is then the first two terms of a perturbation expansion of $1/K_r$.

At high temperature near the phase transition where $K_r\to 0$, the integral becomes divergent, and the perturbation series breaks down. To evaluate Eq. (6) in this regime I use the vortex-core rescaling technique of José *et al.*[17] The integral is divided into two parts: The first part ranges over the small interval from a_0 to ba_0, with $b-1\cong\ln b\ll 1$, and the second part from ba_0 to ∞. Changing variables in the second part $R\to bR$ results in

$$\frac{1}{K_r}=\frac{1}{K}+Ay\ln b+\int_{a_0}^{\infty}\frac{dR}{a_0}Ab^7\left(\frac{R}{a_0}\right)^6y^{bR/a_0}\exp\left(-2\pi^2Kb\frac{R}{a_0}\ln\frac{Rb}{a_0}\right). \tag{7}$$

However, in this increase of scale size we require that Eq. (7) have the same functional form as (6). This is accomplished by our defining new variables at the increased scale,

$$1/K'=(1/b)[(1/K)+Ay\ln b], \tag{8a}$$

$$y'=y^be^{-2\pi^2Kb\ln b}, \tag{8b}$$

$$A'=Ab^6, \tag{8c}$$

$$K_r'=bK_r, \tag{8d}$$

where higher-order terms in y have been neglected. The scaling of g is determined by our taking into account the higher-order correction terms of Eqs. (1) and (2); since they are only a function of a/R, g has to transform as $g'=bg$.

By successive repetition of this transformation, the scaling relations can be put in differential form,[17,18] if we set $dl=\ln b$,

$$\frac{\partial(1/K)}{\partial l}=-\frac{1}{K}+A_0y, \tag{9a}$$

$$\partial y/\partial l=[6-2\pi^2K(\ln\sqrt{gK}+C+1)]y, \tag{9b}$$

$$g=g_0e^l, \tag{9c}$$

$$K_r(K(l),y(l),g(l))=K_r(K_0,y_0,g_0)e^l, \tag{9d}$$

where y has been redefined to absorb the variation of A, $e^{6l}y\to y$, and where $A_0=\pi^5/12$, $y_0=\exp(-2\pi^2K_0C)$, and $g_0=1/K_0$, with K_0 being the initial value of K at the scale size a_0. Equation (9d) explicitly shows that this model satisfies the Josephson hyperscaling relation,[18] i.e., that the specific-heat exponent α and the superfluid-density exponent ν are related by $\nu=(2-\alpha)/3$. Since, for $T<T_c$, the parameter $y(l)\to 0$ as $l\to\infty$, the observable superfluid density is obtained from Eq. (9d),

$$K_r(K_0,y_0,g_0)=\lim_{l\to\infty}K_r(K(l),y(l),g(l))e^{-l}=\lim_{l\to\infty}K(l)e^{-l}. \tag{10}$$

Equations (9a)–(9d) have a simple physical interpretation in terms of a vortex-ring energy which is screened by rings of smaller size. Integration of Eqs. (9a) and (9b) to a finite length l and setting $l=\ln(R/a_0)$ and $K_r=Ke^{-l}$ yields an integral equation for K_r similar to Eq. (5),

$$\frac{1}{K_r}=\frac{1}{K_0}+A_0\int_{a_0}^{R}\frac{dR}{a_0}\left(\frac{R}{a_0}\right)^6e^{-U(R)}, \tag{11}$$

except that now $U(R)$ is a screened energy,

$$U(R)=2\pi^2\int_{a_0}^{R}K_r\left[\ln\frac{R}{a_r}+C+1\right]\frac{dR}{a_0}$$
$$=\int_{a_0}^{R}\frac{1}{\mu}\frac{\delta U_0}{\delta R}dR. \qquad (12)$$

The original energy U_0 is reduced by the scale-dependent permeability in the same manner as in the original arguments of Kosterlitz and Thouless.[10] The parameter $a_r=a_0(K_0/K_r)^{1/2}$ can be interpreted as an effective core radius which increases as the transition is approached. The "real" core size is still a_0; one can probably think of the core of a large ring as a region of increased density of rings of smaller size, each of whose core, in turn, is composed of still smaller rings. This scale-dependent effective core size differs from a conjecture by Nelson and Toner[6] that the core size could exceed the ring diameter, making the ring indistinguishable from a fluctuating "blob" of normal fluid. That does not occur in the present model because the effective core is made up only of rings of smaller size.

The renormalized superfluid density K_r can be calculated numerically from either Eq. (9) or Eq. (11) with standard recursion techniques. The model yields a power-law phase transition where K_r drops to zero as $K_r=(K_0-K_{0c})^\nu$, with fits giving $K_{0c}=0.24846$ and $\nu=0.526$.[19] The solid curve in Fig. 1 shows this behavior, plotted in the form of ρ_s/ρ_s^0 vs T/T_c. As T_c is approached it becomes necessary to carry the iterations to increasingly large values of R before ρ_s stops changing. The critical point $\rho_s=0$ corresponds to the excitation of rings of infinite diameter,[3–5] made possible by the screening effects. The correlation length,[20]

$$\xi=\lim_{l\to\infty}a_0/K_r=m_4^2k_BT/\hbar^2\rho_s,$$

is basically the diameter of the largest ring appreciably excited at a given value of $T-T_c$.

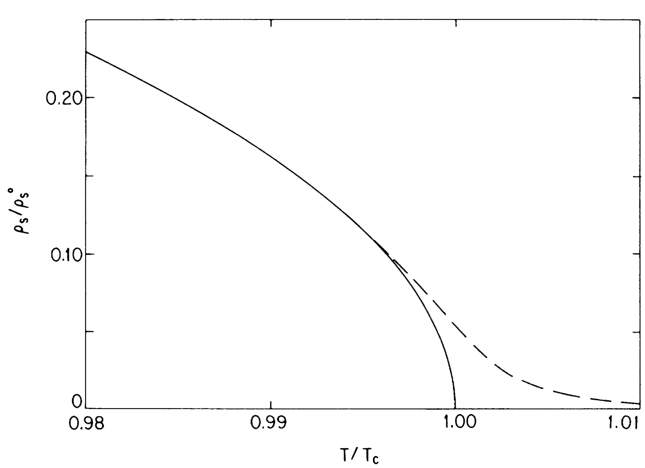

FIG. 1. Superfluid density near T_c, plotted as ρ_s/ρ_s^0 ($=K_r/K_0$) vs T/T_c ($=K_{0c}/K_0$), calculated with Eqs. (9). The solid curve is the result of iteration to $R/a_0\to\infty$, while the dashed curve is a finite-size case of limiting the iteration to $R/a_0=10$.

The real-space nature of this model makes the treatment of finite-size effects particularly simple, requiring only a cutoff of the recursion relations at the finite scale. The dashed curve in Fig. 1 shows the effect of limiting the maximum ring size to $R/a_0=10$. This characteristic finite-size "tail" has been observed in experiments with ^{4}He in confined geometries.[21]

The form of the recursion relations of Eqs. (9) is quite similar to the $2+\epsilon$ expansion of Fisher and Nelson,[22] also derived from a basis of interacting vortices. The flow diagram for y vs $1/K$ is very similar to their Fig. 4, although the fixed-point values are different. The additional terms in Eq. (9b) such as $\ln\sqrt{gK}$ which appear in our model would be higher order in ϵ in their expansion and hence neglected. It is just these terms which give rise to the effective core-size variation in the 3D case ($\epsilon=1$), and it is also these terms which increase the value of the superfluid exponent ν above the result $\nu=0.50$ of the $2+\epsilon$ calculation.

The fact that the superfluid-density exponent in the above calculation does not match the known value $\nu=0.67$ is certainly an indication that the model is not a complete description of the transition. The use of strictly circular rings is a major simplifying assumption that allows the relatively uncomplicated form of Eq. (5). The superposition of a thermal distribution of circular rings generates a mass of tangled vorticity, but this is still not completely equivalent to the more realistic case of distorted, noncircular rings.[4,5] It should be remarked that the derivation of Eq. (5) is quite phenomenological, following the original method of Kosterlitz and Thouless, and more rigorous approaches to it are needed.[23]

A scenario of the superfluid λ transition can be postulated, based on this model and the Monte Carlo simulations. At low temperatures rings are energetically unfavorable [Eq. (1)] because the superfluid density is large. The phonons and rotons are the dominant excitations, giving rise to the temperature-dependent ρ_s^0 according to the Landau two-fluid mode. At higher temperatures approaching T_λ, ρ_s^0 falls to such a small value that it then becomes favorable to excite rings of radius larger than a_0. These rings lower the superfluid density to the renormalized value ρ_s, which makes it possible to excite even larger rings, which then further lower ρ_s, exciting even larger rings, and so on; this finally results in $\rho_s\equiv 0$ at $T=T_\lambda$. Vortex excitations, with energy proportional to ρ_s, are the only soft modes in liquid helium. It is the phase fluctuations of the vortices which generate the amplitude fluctuations [through, e.g., Eq. (5)] that drive the order parameter to zero at the transition.

In summary, a vortex-ring model of the superfluid transition has been constructed. Screening effects of the vortices give rise to a phase transition, as also observed in the Monte Carlo simulations of the 3D XY model. The real-space identification of the vortex fluctuations responsible for the transition should be useful for insight into other calculations such as finite-size effects, boundary-value problems, and the dynamic properties of the transition.

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