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Self-Dual Fields as Charge-Density Solitons

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Two-dimensional Poincaré-invariant self-dual fields are consistently quantized. A fermionic formulation is shown to be equivalent to a nonlocal bosonic one. Self-dual boson fields are solitons describing a charge-density wave of paired fermions.

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There exists wide interest in quantization of two-dimensional self-dual fields, i.e., fields Ψ that satisfy $(g^{\mu\alpha}+\epsilon^{\mu\alpha})\partial_\alpha\Psi=0$, or $\dot{\Psi}=\Psi'$.[1] (Overdot and prime signify differentiation with respect to time t and space x.) However, the canonical/quantal formulation for this simplest of all field dynamics appears to present great difficulties, and elaborate quantization procedures have been invoked, e.g., Becchi-Rouet-Stora-Tyutin quantization, first- and second-class constraints, auxiliary fields, etc. Moreover, it has been alleged that a single self-dual scalar field cannot be quantized. We offer here various observations on this problem, which we encountered in the construction of field-theoretic representations for two-dimensional conformal transformations.[2] Our formulations involve spatially local and nonlocal actions. They are invariant against the Poincaré group of transformations, which spontaneously contracts to a two-parameter group. One version is manifestly invariant.

Consider a boson field χ which satisfies the equal-time commutation relation

$$[\chi(x),\chi(y)]=i\delta'(x-y) \tag{1}$$

with dynamics governed by a local Hamiltonian H, a spatial integral of a Hamiltonian density $\mathcal{H}$,

$$H=\frac{1}{2}\int dx\,\chi^2(x). \tag{2}$$

(We suppress the common time argument of all operators.) Clearly the field satisfies a self-dual equation of motion,

$$\dot{\chi}=i[H,\chi]=\chi'. \tag{3}$$

This theory is obtained from the Lagrangean

$$L=\frac{1}{4}\int dx\,dy\,\chi(x)\epsilon(x-y)\dot{\chi}(y)-\frac{1}{2}\int dx\,\chi^2(x) \tag{4}$$

whose Euler-Lagrange equations imply (3), and canonical quantization gives (1). (Quantization of first-order Lagrangeans is reviewed in the Appendix.) In fact, the Euler-Lagrange equations read

$$\chi(x)=\frac{1}{2}\int dy\,\epsilon(x-y)\dot{\chi}(y), \tag{5}$$

and so they imply, in addition to (3), a boundary condition consistent with (1), $\chi(+\infty)=-\chi(-\infty)$, which indicates that soliton excitations play a role. Indeed it will emerge that χ can be considered a charge-density soliton.

To study the Poincaré group of transformations, we first consider time and space translations. Infinitesimal time translations of the field are as always $\delta_T\chi=\dot{\chi}$. The Lagrangean changes by a total time derivative $\delta_T L=\dot{L}$, and Noether's theorem gives the energy constant of motion, which of course is the Hamiltonian (2):

$$\int dx\frac{\delta L}{\delta\dot{\chi}(x)}\delta_T\chi(x)-L=\frac{1}{2}\int dx\,\chi^2(x)=H. \tag{6}$$

Space translations $\delta_S\chi = -\chi'$ leave L invariant, and the total momentum constant of motion, according to Noether's theorem,

$$P=\int dx \frac{\delta L}{\delta\dot{\chi}(x)}\delta_S\chi(x) = -H \tag{7}$$

is equal to the negative energy. It is important to appreciate that this equality holds on the entire configuration space, not just for solutions to (3), which depend only on $t+x$ and describe left-propagating massless particles with momentum opposite to their energy. The equation of motion is *not* used to establish (7).

Because of (7), the Poincaré algebra of H, P, and the Lorentz generator M ($[H,P]=0$, $[M,H]=iP$, $[M,P]=iH$) contracts to $[H,M]=iH$. Moreover, the Lorentz generator, $M=\int dx\, x\mathcal{H}(x)-tP$, becomes in our model

$$M=\frac{1}{2}\int dx (t+x)\chi^2(x). \tag{8}$$

The Lorentz transformation rule for the field χ is therefore

$$\delta_L\chi(x)=i[M,\chi(x)]=\chi(x)+(t+x)\chi'(x). \tag{9}$$

This leaves the Lagrangean invariant and Noether's theorem reproduces M.

The Lagrangean is also invariant against the infinite two-component two-dimensional conformal group, which contracts to one component, with infinitesimal transformation law for the field

$$\delta\chi(x)=[f(t+x)\chi(x)]' \tag{10}$$

involving an arbitrary function f. Noether's theorem gives generators

$$Q_f=\frac{1}{2}\int dx\, f(t+x)\chi^2(x) \tag{11}$$

that satisfy the infinite conformal algebra.

The Lorentz transformation law (9) is unconventional, although it can be put in familiar form by use of the equation of motion:

$$\bar{\delta}_L\chi(x)=\chi(x)+t\chi'(x)+x\dot{\chi}(x). \tag{12}$$

Nevertheless, the generator that effects (9) satisfies (contracted) Lie algebra, without use of equations of motion. The more conventional transformation (12) maps solutions into solutions, but does not leave the action $I=\int dt\, L$ invariant. The transformed action, however, has the same critical points. This is seen as follows. Write I as

$$I=\frac{1}{2}\int d^2z_1 d^2z_2 \chi(z_1)K(z_1-z_2)\chi(z_2), \tag{13a}$$

where z is the two vector (t,x) and the kernel K is

$$K(z)=\tfrac{1}{2}(\partial_t-\partial_x)\delta(t)\epsilon(x). \tag{13b}$$

When the field is Lorentz transformed according to (12), which in finite form reads

$$\bar{\chi}(z)=e^{\lambda}\chi(\Lambda z),\quad \Lambda z=\begin{pmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{pmatrix}\begin{pmatrix}t\\ x\end{pmatrix}, \tag{14}$$

the transformed action acquires a new kernel:

$$\bar{K}(z_1-z_2)=e^{-2\lambda}K(\Lambda(z_1-z_2)),$$
$$K(\Lambda z)=\tfrac{1}{2}e^{-\lambda}(\partial_t-\partial_x)\delta(t\cosh\lambda-x\sinh\lambda)\epsilon(x). \tag{15}$$

It is easy to see that K and $\bar{K}$ possess the same zero modes. Because the action changes under the transformation, Noether's theorem cannot be used to derive a conserved generator—but we do not need one, since (8) and (9) do the job.

The remaining quantization is uneventful. $\chi(x)$ may be decomposed (at $t=0$) as

$$\chi(x)=-i\int_0^\infty dk\left(\frac{k}{2\pi}\right)^{1/2}[a(k)e^{-ikx}-a^\dagger(k)e^{ikx}]. \tag{16}$$

With

$$[a(k),a^\dagger(k')]=\delta(k-k'), \tag{17}$$

(1) is reproduced and $a^\dagger(k)$ creates a left-moving massless particle with energy $k\geq 0$ and momentum $-k$. But as we shall see, these are not the elementary excitations. Normal ordering produces no surprises, other than the usual extension in the conformal algebra:

$$[\tfrac{1}{2}:\chi^2(x):,\tfrac{1}{2}:\chi^2(y):]=\tfrac{1}{2}i[:\chi^2(x):+:\chi^2(y):]\delta'(x-y)-(i/24\pi)\delta'''(x-y). \tag{18}$$

A local action is obtained by a redefinition of the dynamical variable[3]:

$$\phi(x)=\frac{1}{2}\int dy\,\epsilon(x-y)\chi(y),\quad \phi'(x)=\chi(x). \tag{19}$$

Substitution of (19) in (4) gives a local Lagrangean, with density

$$\mathcal{L}=\tfrac{1}{2}\phi'\dot{\phi}-\tfrac{1}{2}\phi'^2, \tag{20}$$

and the nonlocality hidden in (19) reappears in the equal-time commutator. From (1) and (19) or from canonical quantization of (20) one gets

$$[\phi(x),\phi(y)]=-\tfrac{1}{2}i\epsilon(x-y). \tag{21}$$

While the Hamiltonian formulation is, by virtue of the fixed-time definition (19), equivalent to the previous and leads to the self-dual equation for ϕ, the Euler-Lagrange equations that follow from $\mathcal{L}$ are $\dot{\phi}'=\phi''$, i.e., ϕ' is self-dual, as follows also from (19). Hence the formulation in terms of the local field χ with nonlocal dynamics is preferable to the use of the nonlocal field ϕ with local dynamics.

A completely local and manifestly Poincaré-invariant model may be constructed in terms of fermionic variables that expose the elementary excitations and are appropriate in view of the soliton boundary conditions on χ. In two dimensions, the local, Poincaré-invariant fermion Lagrange density $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ reduces for Weyl fermions, because these are described by a one-component spinor u:

$$\mathcal{L}_{\mathrm{W}} = iu^\dagger(\dot{u} - u'). \tag{22}$$

$\mathcal{L}_{\mathrm{W}}$ transforms as a scalar under the conventional Lorentz transformation,

$$\delta_{\mathrm{L}} u(x) = \tfrac{1}{2}u(x) + tu'(x) + x\dot{u}(x) \tag{23}$$

and obviously gives a self-dual Euler-Lagrange equation for u.

Canonical quantization of (22) leads to the anticommutator

$$\{u^\dagger(x), u(y)\} = \delta(x-y) \tag{24}$$

and the self-duality equation emerges canonically from the Hamiltonian,

$$H_{\mathrm{W}} = i\int dx\, u^\dagger(x)u'(x), \tag{25}$$

which again is the negative momentum. The excitations are left-moving positive and negatively charged particles with energy equal to the negative momentum. The charge density ρ is measured by $\frac{1}{2}[u^\dagger, u]$.

In fact the fermionic formulation is equivalent to the earlier bosonic one, when we identify

$$u(x) = \lim_{m\to 0}\left(\frac{m}{2\pi}\right)^{1/2} :\exp[-i(2\pi)^{1/2}\phi(x)]:,$$
$$u^\dagger(x) = \lim_{m\to 0}\left(\frac{m}{2\pi}\right)^{1/2} :\exp[i(2\pi)^{1/2}\phi(x)]:, \tag{26}$$

with m an infrared regulator needed to define the normal-ordered exponential (see below). The proof is standard,[4] and follows the following steps. Define

$$(2\pi)^{1/2}\phi(x) = \theta(x) + \theta^\dagger(x), \tag{27}$$

where θ is the positive-frequency (annihilation) part in a decomposition of ϕ similar to (16),

$$\theta(x) = \int_0^\infty \frac{dk}{\sqrt{k}} a(k) e^{-ikx}. \tag{28}$$

Thus

$$u(x) = \lim_{m\to 0}\left(\frac{m}{2\pi}\right)^{1/2} e^{-i\theta^\dagger(x)} e^{-i\theta(x)}. \tag{29}$$

The commutator between $\theta(x)$ and $\theta^\dagger(x)$,

$$C(x-y) = [\theta(x), \theta^\dagger(y)] = \int_0^\infty \frac{dk}{k} e^{-ik(x-y)}, \tag{30a}$$

is regulated in the infrared by a mass m, and is taken as

$$C(x) = -\ln m|x| - \tfrac{1}{2}i\pi\epsilon(x)$$
$$= -\ln m(x - i\epsilon) - \tfrac{1}{2}i\pi. \tag{30b}$$

The first equality in (30b) exhibits the fact that $\mathrm{Im}C(x)$ is determined by the commutator (21) of ϕ, while the second puts into evidence the "$i\epsilon$" ultraviolet regularization needed in the half-line integral (30a). Repeated use of the re-normal-ordering formulas

$$e^{i\theta(x)}e^{i\theta^\dagger(y)} = e^{i\theta^\dagger(y)}e^{i\theta(x)}e^{-[\theta(x),\theta^\dagger(y)]} = im(x-y-i\epsilon)e^{i\theta^\dagger(y)}e^{i\theta(x)},$$
$$e^{i\theta(x)}e^{-i\theta^\dagger(y)} = e^{-i\theta^\dagger(y)}e^{i\theta(x)}e^{[\theta(x),\theta^\dagger(y)]} = [im(x-y-i\epsilon)]^{-1}e^{-i\theta^\dagger(y)}e^{i\theta(x)}, \tag{31}$$

allows deduction of the following:

I. For $x \neq y$, u and $u^\dagger$ are anticommutating variables.

II. $\langle 0|\{u(x),u(y)\}|0\rangle = 0$, $\langle 0|\{u^\dagger(x),u(y)\}|0\rangle = \delta(x-y)$.

III. $\frac{1}{4}[u^\dagger(x+\frac{1}{2}\eta), u(x-\frac{1}{2}\eta)] + \frac{1}{4}[u^\dagger(x-\frac{1}{2}\eta), u(x+\frac{1}{2}\eta)] = (2\pi)^{1/2}\chi(x) + O(\eta)$.

IV. $\frac{1}{2}i[u^\dagger(x+\frac{1}{2}\eta)u'(x-\frac{1}{2}\eta) - u^{\dagger\prime}(x-\frac{1}{2}\eta)u^\dagger(x+\frac{1}{2}\eta)] = (4\pi)^{-1}[(\eta+i\epsilon)^{-2} + (\eta-i\epsilon)^{-2}] + \frac{1}{2}:\chi^2: + O(\eta)$.

Results I and II show that (26) and (29) are valid representations of fermions by bosons ϕ or χ, while the remaining two results imply that the Hamiltonian H_{W} (25) for the Fermi field u, when normal ordered, coincides with the normal-ordered Hamiltonian H for the Bose field χ or ϕ. This follows from III which in the limit $\eta \to 0$ expresses the fermion charge density ρ as $(2\pi)^{-1/2}\chi$, and then the Sugawara-Sommerfield construction equates the respective Hamiltonians.[5] Alternatively IV gives the result directly.

From III, we also deduce that χ is a charged-density soliton, and its excitations are charge-neutral pairs of elementary fermionic excitations.

The conclusion is that a single self-dual Bose field can be quantized from a local or nonlocal action, which is invariant under Poincaré transformations that are of unconventional form because of the contraction of the Poincaré group. The fermion formulation, which is manifestly Poincaré invariant, exposes the elementary excitations as fermionic, and accounts for the necessary nonlocality of the composite bosons: They are now seen as charge-density solitons. Moreover, we conclude that the minimal self-dual field is the Majorana-Weyl fermion, also described by (22), (24), and (25), but with Hermitean field $u^{\dagger}=u$. It is not known whether this single charge-neutral excitation has a bosonic formulation. If it does, presumably a much more severe nonlocality than that of χ or ϕ is involved.

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Appendix.— We review the quantization of a first-order particle Lagrangean,

$$L=\tfrac{1}{2}q_iC_{ij}\dot{q}_j-V(q), \tag{A1}$$

where C is a constant matrix which may be taken antisymmetric, since a symmetric part leads to a total derivative in L. We also take C to possess an inverse, so that L is nonsingular. The equation of motion reads

$$\dot{q}_i=C_{ij}^{-1}\,\partial V(q)/\partial q_j. \tag{A2}$$

Commutation relations are defined to reproduce (A2) from the Hamiltonian, which here is V. Hamiltonian equations of motion

$$\begin{aligned}\dot{q}_i&=i[H,q_i]=i[V(q),q_i]\\&=[\partial V(q)/\partial q_j]i[q_j,q_i]\end{aligned} \tag{A3}$$

coincide with (A2) provided

$$[q_i,q_j]=iC_{ij}^{-1}. \tag{A4}$$

The extension to boson and fermion field theories is obvious and justifies the commutators in the text.

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