Lower Critical Dimension for the Random-Field Ising Model

J. Bricmont
Institut de Physique Théorique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

and

A. Kupiainen
Research Institute for Theoretical Physics, University of Helsinki, SF-00170 Helsinki 17, Finland
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We show that the Ising model in three dimensions with a small random magnetic field has two phases at low temperatures, i.e., that its lower critical dimension is at most 2. This is shown by our devising an exact renormalization-group flow which takes the theory to the zero-temperature zero-field fixed point.

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The random-field Ising model (RFIM)[1] has been introduced to describe several physical systems, e.g., dilute anisotropic antiferromagnets in a uniform field. Since the seminal work of Imry and Ma,[2] a (sometimes intense) discussion has been going on in the literature about the lower critical dimension d_l of this model. d_l is the dimension above which there is an ordered phase at low temperatures. We outline here a rigorous proof (for full details, see Bricmont and Kupiainen[3]) of $d_l \leq 2$, i.e., we show that, in $d \geq 3$ dimensions, there are two ordered phases at low temperature and small disorder. In other words, the Ising phase transition is stable with respect to a small random perturbation.

The main heuristic arguments in favor of $d_l = 2$ were given by Imry and Ma and were put on a more solid basis by Fisher, Fröhlich, and Spencer[4] and Chalker.[5] Subsequently, Imbrie[6] proved that, if $d = 3$, the ground state is ordered with probability 1, which gave very strong support to the conjecture $d_l = 2$. However, some doubts still subsisted as to whether there is ordering at finite temperature in $d = 3$ dimensions. Indeed, one of the arguments leading to $d_l = 3$ was based on the ideas of Parisi and Sourlas on dimensional reduction.[7-9] This predicts that a random system in d dimensions behaves like the corresponding deterministic one in $d-2$ dimensions. However, for $d = 1$, the Ising model is ordered at zero temperature, but not at any nonzero T. Our result rules out a similar behavior for the $d = 3$ RFIM. Indeed, our analysis shows that, unlike in the one-dimensional Ising model, the T here is an irrelevant variable.

Another line of thought that supported the (wrong) conjecture $d_l = 3$ was based on an estimate of the size of the fluctuations of an interface separating the + and the − phases. This estimate was derived from a $5-d$ expansion, whose results were also the subject of some controversy. Depending on how this expansion was defined, the width of the interface of size L was found to diverge as[10-12] $L^{(5-d)/3}$ or as[13-16] $L^{(5-d)/2}$.

We cannot completely resolve this controversy, but we can certainly rule out $(5-d)/2$ for $d = 3$. Indeed, in $d = 3$ dimensions, this law means that the interface diverges like its length, which is akin to say that there is no phase transition. It is possible that an extension of our methods would actually give $L^{2/3}$ as an upper bound for this divergence, but we have not pursued this. On the other hand, for $d \geq 4$, it should be rather simple (but we have not checked the details) to prove that the interface does not diverge at all, at low T and small disorder. It would be similar to the deterministic model in $d \geq 3$ dimensions (we note that, as far as the lower critical dimensions are concerned, the random model is analogous to the deterministic one in $d-1$ rather than $d-2$ dimensions). However, for $d = 4$, the $(5-d)/3$ power law may well be correct, as it was meant to be, above the roughening transition.

We next turn to the precise results. At each site of Z^3 we have an Ising spin, and the Hamiltonian is

$$H = -\sum(\sigma_x \sigma_y - 1) - \sum h_x \sigma_x, \tag{1}$$

where the first sum runs over nearest neighbors, and h_x are Gaussian independent, identically distributed random variables, of mean zero and variance ϵ^2 (our arguments can be extended to more general distributions). Let +,− denote the distributions obtained as thermodynamical limits of Gibbs distributions in finite volumes with as boundary conditions all spins + or − (these exist for any realizations of the h). Then if T and ϵ^2 are small enough, and $d \geq 3$,

$$\langle \sigma_x \rangle^{\pm} = \pm 1 + O(e^{-\beta}), \tag{2}$$

with probability $1 - e^{-1/\epsilon^2}$. From this, one obtains easily probability-1 statements. For example, the h average of the magnetization satisfies

$$\overline{m} = \pm 1 + O(e^{-\beta} + e^{-1/\epsilon^2}),$$

and, by ergodicity,

$$\overline{m} = \lim_{\Lambda \to Z^3} (1/\Lambda) \sum_x \langle \sigma_x \rangle$$

with probability 1. Thus, there are, as in the deterministic model at $h=0$, two distinct phases.

We will now give some ideas of the proof. It is based on the Peierls method of contours combined with a recent renormalization-group (RG) theory of first-order phase transitions.[16] Let us recall first the low-temperature contour representation. Choose a boundary condition. Then each spin configuration is in one-to-one correspondence with a family of contours γ, i.e., γ are closed surfaces separating + and − spins. Each γ carries a weight $\rho(\gamma) \simeq e^{-\beta|\gamma|}$ and the partition function is given by $e^{\beta(h,V)}Z$ with

$$Z = \sum_{\{\gamma\}} \prod_\gamma \rho(\gamma) e^{\beta(H,V_-)}, \tag{3}$$

where we separated the ($\epsilon=0$) ground-state energy and used notation $(h,V)=\sum_{x\in V} h_x$. V_- is the region of minus spins determined by $\{\gamma\}$ and $H=-2h$.

If $h=0$, the convergence of the sum is standard, the entropy of a contour being $e^{O(|\gamma|)}$. For $h\neq 0$, the field enclosed by a contour may give an $e^{O(\mathrm{vol}\gamma)}$ contribution which, for large enough γ, will dominate $\rho(\gamma)$. This, of course, happens if there is a large set of h of the wrong sign, forcing σ to the false (−) vacuum, and so creating a contour.

Imry and Ma noted that this should be quite improbable: e.g., for a cubic contour of volume $\mathrm{vol}\gamma = L^3$, the field $H(\gamma)=[H,\mathrm{Int}(\gamma)]$, being a sum of L^3 independent random variables, has variance $L^3\epsilon^2$ and thus

$$\mathrm{Prob}[H(\gamma) > |\gamma|] \sim \exp[-O(L/\epsilon^2)],$$

since $|\gamma| \simeq L^2$. Thus the dangerous contours become more unlikely the larger they are.

This argument has two well-known problems. The first is that there are $e^{O(L^2)}$ contours of size $|\gamma|=L^2$, and this number dominates the above probability. This problem was solved in Refs. 4 and 5 with use of a coarse-graining argument. The second problem is that the Imry-Ma argument ignores the possibility of contours within the contour we are studying. This is serious, since we wish to show the unlikeliness of contours and should not assume it. Of course, such contours should renormalize the field within γ, and we should try to show that the renormalized field remains essentially the same as the original one.

Since the contours occur on all length scales, a multiscale renormalization-group analysis presents itself naturally. As usual, our renormalization-group transformation (RGT) consists of two steps. First, one integrates out the short-distance fluctuations, i.e., small contours, and then rescales and relabels everything in order to get a new system, almost of the form of the original one, but with new (effective) parameters. Iteration of the RGT then produces a flow of the effective parameters.

Thus, let us "sum out" of (3) contours of linear size $\leq L$, an arbitrary parameter ~ 1. However, we will do this only in regions of the lattice where the field is small enough not to dominate the contour activity. We define the large-field region $D=\{x: |H_x| > \delta\}$. Then, provided that δ is small, we may compute the free energies $F^\pm$ of the small contours outside D in the $\pm$ seas determined by the large contours by a convergent low-temperature expansion. F can be interpreted as a contribution to a new effective h produced by the small contours:

$$F^\pm = \sum_x \delta h_x^\pm,$$

where the δh are almost-local functions of the H, bounded by $O(e^{-\beta})$.

The second part of the RGT consists of regrouping the large contours on a blocked lattice so as to produce again upon rescaling distances a system with contours on all scales, only with a renormalized field and activities. Cover the lattice with disjoint cubes of side L. Given a family of contours $\Gamma=\{\gamma_l\}$, we block them on scale L, i.e., we define the new (blocked) contours as connected components of the set of L-cubes intersected by Γ and D. Observe that we include the large-field region into the contours. This is done mostly for convenience. The new contours have an activity $\rho'(\gamma')$ which is just the sum of the activities of the old contours, and the parts of D, which produce γ' under blocking. The new contours live on the unit lattice again.

The renormalized field H' is

$$H'_x = L^{-2}\sum_y (H_y + \delta h_y^- - \delta h_y^+), \tag{4}$$

with y in the L^3 block centered at Lx. The peculiar scaling dimension of H will be explained below. The upshot of these manipulations is that Z may be written as $e^{\beta(\delta h^+,V)}Z'$ (we pulled out the renormalization of the ground-state energy due to small contours) with

$$Z' = \sum_{\{\gamma'\}} \prod_{\gamma'} \rho'(\gamma') e^{\beta'(H',V^-)}, \tag{5}$$

where the renormalized β is

$$\beta' = L^2\beta$$

[actually, (5) should also contain a small interaction between the contours, but this is left out from the present qualitative discussion].

The renormalized partition function is of the same form as the original one which we started with; thus we may iterate the above procedure. Note that the new field H' may be as large as $L\delta$ and upon iteration would eventually dominate even the small contours. Thus we define the new large-field region D' by adding to the one obtained by blocking D the region where H' is bigger than δ, and repeat the above procedure.

How will the activities ρ' and the distribution of the renormalization field H' flow? Consider H'. We started with independent H_x's. Without the $\delta h^\pm$ in (4), H' is a sum of L^3 independent random variables and thus has variance $L^3L^{-4}\epsilon^2 = L^{-1}\epsilon^2$. The correction $\delta h^\pm$ is $O(e^{-\beta})$ and almost local. One easily shows that H' have variance $L^{-1}\epsilon^2 + O(e^{-\beta})$ and are almost independent. Thus the small contours produce a negligible contribution to the "no contours within contours" picture, and the variance of the random field is an irrelevant variable.

Turning to the flow of ρ', we first note that, even ignoring D, the new contours can be quite different from what one usually means by a contour. In general, they need not necessarily flip the signs (e.g., block two nearby contours together) or have an interior (block a long thin contour). However, in order to iterate the RGT, we need only to have a bound for the small contours which dominates the small H in their interior. So the following bound, which will be proven inductively, will be sufficient for the activities:

$$\rho(\gamma) < \exp\left[-\beta|\partial \mathrm{Int}\gamma| - \tilde{\beta}|\gamma \backslash D| + \beta \sum_{x \in \gamma \cap D} N_x\right], \tag{6}$$

with $\beta, \tilde{\beta}$ running as

$$\beta_n = L^{2n}\beta, \quad \tilde{\beta}_n = L^n \tilde{\beta}.$$

It is easy to understand the various parts in (6). The boundary of the interior of a contour, $\partial \mathrm{Int}\gamma$, is a two-dimensional region and thus scales upon blocking as L^2. This explains β and our choice of the scaling in (4): We wanted to have the same β in front of the geometrical factor and of the H. The one-dimensional parts of the contour scale as L (since the contours are connected). This explains $\tilde{\beta}$ (note that on D we need not have any contour strength).

N is an upper bound on the contributions from the D region, i.e., all the large H^n's collected during the iteration. Obviously after the first step, $N_x = L^{-2}\sum |h_y|$, with y in the block of Lx, will do. And iteratively,

$$N_x' = L^{-2}\sum N_y + |H_x'|\chi(H_x'),$$

where χ is 1 if $|H'| > \delta$ and 0 otherwise.

We have chosen so far to ignore the renormalization of the large fields altogether. This cannot work as such. Even an initial D consisting of a single point would upon blocking comprise eventually the full volume of the box and we obviously would lose all control of the system. To see what to do, consider the physical effect of a large field: It may force a spin to be in the direction opposite to the global magnetization induced by the boundary condition. That is, it may create contours. However, they can be created only up to a certain length scale depending on the size of the large field. For example, if $h_x = L^k$ on the first scale and all the fields nearby are small on all scales, we may remove this h after k steps from D, since any contour that has survived the integration of the small contours that long must have had, on the original scale, a size at least L^k and thus cannot be dominated by this large field. This procedure may be formulated in terms of the N variables; as N_x is small enough such an x is removed from D, i.e., we exponentiate small contours comprising this point too. The contributions of the small contours including such N's to the H' of (4) are again negligible.

The RGT we have outlined above is defined for an arbitrary h configuration. If we want to prove properties of magnetization, etc., we need to know that the D regions, where we have very little control on how the activities behave, are very improbable. This amounts to showing that in the large field region, too, the random field tends to the zero-variance fixed point. In fact, the probability of there being an N at all at any scale is extremely small: First, note that the probability of $|H| > \delta$ is $e^{-\delta^2/\epsilon^2}$ and becomes smaller upon iteration, since ϵ runs. We prove that for $N > 1$, say,

$$\mathrm{Prob}(N_x > N) < e^{-N^2/\epsilon^2} \tag{7}$$

(with the running ϵ).

To summarize, the RG takes our model upon iteration to the zero-temperature zero-field fixed point. The result for the magnetization is now easy. Consider $\langle\sigma_0\rangle$, and the h event $N_0^k = 0$ for all k, i.e., no large effective fields at zero on any scale. With use of (7) it can be shown that this has probability $1 - \sum_k \exp[-O(1/\epsilon_k^2)] \simeq 1 - e^{-1/\epsilon^2}$. Then we just need to see whether there are contours at any scale k around zero, which has weight $\sum_k e^{-\tilde{\beta}_k}$ (more formally, the σ insertion renormalizes additively by $e^{-O(\tilde{\beta}_k)}$). Equation (2) follows.

An interesting open problem concerns the $d=2$ model. It is generally expected that an arbitrary small disorder destroys the phase transition, i.e., there is a unique phase for a typical random field. When analyzed to leading order, our renormalization group shows that the disorder stays constant. We expect higher orders to drive the system towards large disorder, but this appears to be difficult to prove. We have constructed and solved exactly a toy model, which is a hierarchical version of the RFIM. It clearly shows the presence of a spontaneous magnetization in $d \geq 3$ dimensions and its absence in $d=2$ dimensions. However, in $d=2$ dimensions, it exhibits a power-law decay of its correlation functions, which we believe is an artifact of its hierarchical nature, although this is also the subject of some controversy.[17]

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