Two-Loop Chiral Anomalies in Open Superstrings

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(Received 10 August 1987)

We partially analyze the two-loop box and hexagon chiral gauge anomalies in $D=6$ and 10 dimensions for super-Yang-Mills theories. We apply these results to the SO(32) open superstring and show that the leading gauge anomaly vanishes at two-loop order while the subleading gauge anomalies, the mixed anomalies, and the subleading chiral gravitational anomalies may all be canceled with appropriate counterterms. To do this in field theory it is necessary to include a term $B(\mathrm{Tr}F^2)^2$ which is absent at one-loop order in the zero-slope limit of the SO(32) superstring.

PACS numbers: 11.17.+y, 11.30.Rd

In $D=4$ dimensions, renormalizable field theories, such as Yang-Mills fields coupled to chiral fermions, may have chiral anomalies. However, the anomaly one calculates at one-loop order is not corrected in higher orders of perturbation theory, a result first proved by Adler and Bardeen.[1]

In closed-string theories, there exist arguments analogous to the Adler-Bardeen (AB) theorem. For example, in the $E_8 \otimes E_8$ heterotic theory, since there is modular invariance to all orders, anomalies that fail to materialize at one-loop order may not appear in higher-loop (higher-genus) diagrams if they correspond to a breakdown of modular invariance.[2]

In the $E_8 \otimes E_8$ closed-superstring theory, there is only one diagram to consider at each order in perturbation theory; but in the SO(32) open-superstring theory,[3] life is much more complicated. We can also expect no generalized AB argument of the type that exists for the $E_8 \otimes E_8$ superstring, since there is no full modular invariance for the open superstring. If there is an AB theorem, it must involve other arguments or be of the more general type discussed above.

We begin by summarizing the two-loop open-string diagrams that may contribute to chiral anomalies. Let us split these into two classes: (A) diagrams with only open-string internal propagators, and (B) diagrams with closed-string internal propagators. We will be most interested in the hexagon diagrams with six external bosonic legs. (It is both useful and easier to consider the box diagram first, which we do when we come to the calculations.)

Class (A) diagrams can all be represented as disks with two holes cut out. A diagram of this topology can be deformed into a three open-string vertex with three open strings all running to a second three open-string vertex. To make the hexagon, one then attaches the six external legs in all possible combinations to the three internal string propagators. There are seven such possibilities, as we will summarize below. Finally, once the external legs are attached, one must sum over all possible twists in the internal propagators.

The subclasses of diagrams $V^{abc}_{(\mathrm{A})}$ of class (A) can now be summarized by the expression

$$V^{abc}_{(\mathrm{A})} = \sum H^{abc}_{(\mathrm{A})_i} Z^{abc}_{(\mathrm{A})_i}, \tag{1}$$

where i runs over the twists and is the only index that is summed; a, b, and c are the number of external lines attached to each internal open-string propagator. We have factored out the group-theory pieces $H^{abc}_{(\mathrm{A})}$ factors. We have calculated them for all seven (four) subclasses of class (A) diagrams for the hexagon (box) including all possible twists.

Class (B) diagrams can contain at most one internal closed string (and no holes) if they are to be of the same order in perturbation theory as class (A). This is because closed-string diagrams carry a factor of κ, while class (A) diagrams have a factor g^2 (g for each three open-string vertex) beyond the one-loop result. But[4] in type-I superstring theory, $\kappa \propto g^2$. The possibilities for class (B) can also be understood by our considering a disk with two holes. If a tube (closed string) is plugged into one hole on the top of the disk, then a class (B) diagram is made by either our plugging the other end of the tube into the other hole on the top of the disk (making a pot lid) or by our stretching the tube around and plugging it into the second hole from the bottom of the disk. The pot lid is orientable (O); the second case is nonorientable (X). (The second connection of the tube can also be thought of as being connected at a cross cap to make a "Klein pot lid.") Finally, in both cases, one attaches the external lines to the outside rim of the disk.

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We call the two contributions $V_B^{(O)}$ and $V_B^{(X)}$ where $V_B \equiv H_B Z_B$.

At the one-loop level, the Green-Schwarz[3] mechanism depends on the inclusion of an antisymmetric tensor field $B_{\mu\nu}$ in the spectrum; then the tree diagram with an internal B line connecting vertices with two and four external gauge propagators can cancel the nonleading gauge anomaly coming from $(\mathrm{Tr}F^2)(\mathrm{Tr}F^4)$. Since $B_{\mu\nu}$ is in the supergravity multiplet which arises from the closed-string sector of the superstrings, if one cuts this tree graph the closed loop is apparent, and so it is not surprising that the contribution it gives is of the same order as the other one-loop anomaly graphs.

As we will see, at two-loop order the coefficient of the leading $\mathrm{tr}F^6$ term in the hexagon anomaly cannot be zero for any N unless an antisymmetric B field is included. This inclusion of class (B) diagrams allows the cancellation.

We will now summarize the calculations of the various H factors. We consider class (B) first since it is simpler.

Both class (B) diagrams contribute a factor of $N\,\mathrm{tr}\lambda^n \propto N\,\mathrm{tr}F^n$, where n is the number of external gauge bosons, here either 4 or 6; N is the dimension of the fundamental irreducible representation (irrep) $\{\lambda\}$ of SO(N); the trace tr is symmetrized and over the fundamental irrep. (In the sequel traces, Tr will be understood to be symmetrized but over the adjoint irrep.) It is easy to see where the above factors come from. Again, consider a flat disk with two holes and n external lines all attached to the outside boundary. Following "quark" lines, on the outside there is just the trace $\mathrm{tr}\lambda^n$. Since there are no external lines attached to the internal holes, each hole carries a Kronecker δ function in the quark (i.e., fundamental) irrep. Now, stretching the holes out and connecting them, one gets $\delta_b^a \delta_a^b = \delta_a^a = N$. The quarks running around the boundary of the holes must match before we can connect them, i.e., we get N not N^2 as can happen for class (A) diagrams.

For class (A) diagrams we must calculate traces of the form

$$\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\ldots\Lambda^{a_j}\Lambda^{a_j}\Lambda^{b}\Lambda^{a_{j+1}}\ldots\Lambda^{k}\Lambda^{b}\Lambda^{k+1}\ldots\Lambda^{a_n}) \quad (2)$$

for all $j,\, k \le n$. Again n is the number of external legs. Here Λ is in the adjoint irrep of SO(N);

$$\Lambda^a = \delta\lambda^a - \delta\lambda^a + \lambda^a\delta - \lambda^a\delta. \quad (3)$$

The traces (2) are straightforward with use of diagrammatic techniques.[5,6] For the box diagram we find the results given in Table I. The leading $\mathrm{tr}\lambda^4$ term does not vanish for any N for either class (A) or (B) or contributions $H_{(A)}$ or $H_{(B)}$ separately, but does for the sum H when $N=8$ (the same value of N for which the leading one-loop anomaly canceled. The other solution at $N=4$ must be irrelevant, since this choice for N would be one-loop anomalous). It is true that the $\mathrm{tr}\lambda^4$ terms for $H_{(A)}^{400}$ and $H_{(A)}^{310}$ both vanish for $N=8$. This is apparently because $V_{(A)}^{400}$ is a propagator correction for an internal open string, and $V_{(A)}^{310}$ corrects an external vertex. (Note further that $H_{(A)}^{400}$ and $H_{(A)}^{311}$ are proportional to the one-loop anomaly. We find a similar result for $H_{(A)}^{600}$ and $H_{(A)}^{510}$ below.)

TABLE I. Two-loop O(N) box anomaly coefficients in six dimensions.

H^{abc}	$\mathrm{tr}\lambda^4$	$(\mathrm{tr}\lambda^2)^2$
$H_{(A)}^{400}$	$2(N-2)(N-8)$	$6(N-2)$
$H_{(A)}^{310}$	$(N-4)(N-8)$	$3(N-4)$
$H_{(A)}^{220}$	$N^2-10N+32$	$3N-14$
$H_{(A)}^{211}$	$-8(N-4)$	$4(N-4)$
$H_{(A)}$	$4N^2-50N+128$	$8(2N-7)$
$H_{(B)}$	$2N$	0
$H^{(\mathrm{box})}=H_{(A)}+H_{(B)}$	$4(N-4)(N-8)$	$8(2N-7)$

Let us pause a moment to recall the one-loop hexagon anomaly result.[3,6] All type-I chiral superstring theories in $D=10$ have hexagon anomalies for SO(N) gauge groups (gauge part only)

$$A_{\mathrm{hex}}^{1\text{-loop}} = (N-32)\,\mathrm{tr}F^6 + 15\,\mathrm{tr}F^2\,\mathrm{tr}F^4, \quad (4)$$

where F is in the fundamental irrep of SO(N). The leading anomaly piece obviously vanishes at $N=32$. (These anomalies are fatal for other values of N.) What is not so obvious is that the nonleading anomaly can also be removed by the clever trick[3] of adding a piece $B\,\mathrm{Tr}F^4$ (added to the supergravity theory but generated by string theory) to the Lagrangean. The nonleading gravity and mixed anomalies are eliminated in a similar fashion.[3,7] We now return to two loops.

The class (A) contributions to the two-loop hexagon anomaly give the polynomial factors in N summarized in Table II. Class (B) terms and the total $H^{(\mathrm{hex})}$ are also given.

The coefficient of the $\mathrm{tr}\lambda^6 \sim \mathrm{tr}F^6$ term factors. Thus

TABLE II. Two-loop O(N) hexagon anomaly coefficients in ten dimensions.

$H_{(A)}^{abc}$	$\mathrm{tr}\lambda^6$	$\mathrm{tr}\lambda^2\mathrm{tr}\lambda^4$	$(\mathrm{tr}\lambda^2)^3$
$H_{(A)}^{600}$	$2(N-2)(N-32)$	$30(N-2)$	0
$H_{(A)}^{510}$	$(N-4)(N-32)$	$15(N-4)$	0
$H_{(A)}^{420}$	$N^2-22N+128$	$9N-74$	6
$H_{(A)}^{330}$	$N^2-18N+128$	$6(N-13)$	9
$H_{(A)}^{411}$	$-20N+128$	$2(5N-38)$	6
$H_{(A)}^{321}$	$-14N+128$	$7N-82$	9
$H_{(A)}^{222}$	$-12N+128$	$6(N-14)$	10
$H_{(A)}$	$5N^2-190N+896$	$83N-514$	40
$H_{(B)}$	$2N$	0	0
$H^{(\mathrm{hex})}=H_{(A)}+H_{(B)}$	$(5N-28)(N-32)$	$83N-514$	40

we conclude that the leading gauge anomaly vanishes for an SO(32) gauge group. It is also evident that $B_{\mu\nu}$ fields are needed if this leading anomaly coefficient at two loops is to vanish for SO(32).

The group factor $40(\mathrm{tr}\lambda^2)^3$ can be generated by an anomaly $\simeq(\mathrm{tr}F^2)^3$. Such a term cannot appear in one-loop group traces of adjoints but can always appear in two and higher loops. This is the only new tensor invariant available beyond one-loop order in $D=10$. To cancel this nonleading term, a further counterterm of the form $B(\mathrm{tr}F^2)^2$ needs to be introduced. A sketch of the complete two-loop counterterms is given at the end.

A final remark concerning $H^{(\mathrm{hex})}$ is in order. In analogy with previous work in string theory, we assume all seven class (A) and the two class (B) diagrams must be added with equal weight to preserve unitarity.[8,9] It is gratifying that when this is done in D dimensions we get precisely the same $(N-2^{D/2})$ factor to cancel the leading gauge anomaly as was obtained at one loop.

Assuming that the leading gravitational anomaly cancels at one loop and fixing $N=32$ for simplicity, we find (up to an overall normalization) the 12 form, associated with the total two-loop anomaly, must take the form

$$I_{12}=a_1\mathrm{tr}F^4\mathrm{tr}F^2+b_1(\mathrm{tr}F^2)^3+a_2\mathrm{tr}R^4\mathrm{tr}R^2+b_2(\mathrm{tr}R^2)^3 +a_3\mathrm{tr}R^2\mathrm{tr}^4+b_3\mathrm{tr}R^2(\mathrm{tr}F^2)^2+a_4\mathrm{tr}F^2\mathrm{tr}R^4+b_4\mathrm{tr}F^2(\mathrm{tr}R^2)^2, \quad (5)$$

with $a_1=2142$ and $b_1=40$. The consistent anomaly is

$$G\simeq\int I'_{10}, \quad (6)$$

where $I_{12}=dI_{11}$ and $\delta I_{11}=dI'_{10}$ have been used. One must then find a counterterm S_c to add to the action such that $\delta\delta_c=-G$, otherwise the theory is anomalous. The choice of a minimal form,

$$S_c\simeq\int[a_1B\,\mathrm{tr}F^4+b_1B(\mathrm{tr}F^2)^2+a_2B\,\mathrm{tr}R^4+b_2B(\mathrm{tr}R^2)^2+\Delta B\,\mathrm{tr}F^2\mathrm{tr}R^2-\Delta\omega^0_{L3}\omega^0_{Y3}\mathrm{tr}R^2-\Delta\omega^0_{Y3}\omega^0_{L3}\mathrm{tr}F^2], \quad (7)$$

gives the required cancellation provided that, in Eq. (5),

$$a_1=a_3, \quad (8a)$$

$$a_2=a_4, \quad (8b)$$

$$\Delta=(b_3-b_1)=(b_4-b_2). \quad (8c)$$

Since we have computed only a_1 and b_1, we have made only a consistency check. The fact that $b_1\neq0$ at two loops (recall $b_1=0$ at one loop) the counterterm $B(\mathrm{tr}F^2)^2$ appears first only at this order. A similar new term is likely to appear in the O(32) heterotic string, presumably from a different part of the integration region for the double torus. For $E(8)\otimes E(8)$, there is a proportionality between $\mathrm{tr}F^4$ and $(\mathrm{tr}F^2)^2$, so the distinction between counterterms does not exist.

In summary, we have shown there exists no leading gauge anomaly at two-loop order for the SO(32) open superstring and that a new counterterm $B(\mathrm{tr}F^2)^2$ must be added to cancel the anomaly coming from the twelve-form $(\mathrm{tr}F^2)^3$ for which we have calculated the nonvanishing coefficient at two loops. A full analysis of the field-theory part of the two-loop anomalies will undoubtedly be quite difficult but would certainly help clarify the details of the SO(32) open superstring.

We are grateful to L. Alvarez-Gaumé, W. A. Bardeen, and M. T. Vaughn for useful discussions. The work of one of us (T.W.K.) was supported by a U.S. Department of Energy Outstanding Junior Investigator Grant No. DE-FG05-85ER40226, and he thanks R. Blankenbeckler and the Stanford Linear Accelerator Center theory group, and the Aspen Center for Physics for hospitality while this work was in progress. Another of us (P.H.F.) thanks L. Sulak and Boston University theory group for hospitality, and acknowledges the support of his research by the U.S. Department of Energy Grant No. DE-FG05-85ER-40219. The work of another (T.C.Y.) was supported by the U.S. Department of Energy Grant No. DE-FG05-85ER40226.

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