Cancellation of Dilaton Tadpoles and Two-Loop Finiteness in SO(32) Type-I Superstring Theory

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By use of a superspace generalization of the involution technique (method of images), it is shown that dilaton tadpole amplitudes for the world sheets with the topologies of disk and projective plane cancel for the SO(32) type-I superstring. The same method also shows that the amplitudes at the one-loop level, i.e., on the torus, cylinder, Möbius strip, and Klein bottle, vanish after summation over spin structures. The results suggest that the theory is finite up to two loops.

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One of the important problems in string theories is the finiteness of the theories. In fact, recent interest in string theories was largely inspired by the discovery[1] of the cancellation of (one-loop) infinities and anomalies in SO(32) type-I superstring theory. Physically meaningful infinities in string theories can be interpreted as infrared singularities, arising when a massless particle (the dilaton) with the propagator $1/k^2$ from the diagram disappears into the vacuum ($k^\mu=0$).[2,3] [There are other sources of (unphysical) divergences such as those from tachyons, which we neglect here.] Thus it is plausible that the cancellation of physically meaningful infinities (in open-string amplitudes) at L-loop level is implied by the cancellation of dilaton tadpoles at $L-1$ and fewer loops provided there are no other sources of dangerous divergences (and no others are expected).

Recently Douglas and Grinstein[4] have shown that the dilaton tadpoles cancel in purely bosonic open-string theories in 26 dimensions for the group $SO(2^{13})$ at tree level. Subsequently Weinberg[5] has explicitly demonstrated the absence of divergences in the one-loop amplitudes, thus substantiating the above connection. It is expected that the cancellation of infinities in SO(32) superstring theory[1] is also equivalent to that of dilaton tadpoles. Actually the argument can be used in the other way to get dilaton tadpoles from the one-loop amplitudes.[2,5]

It has been pointed out, however, that infinities at the one-loop level suffer from an ambiguity related to regularization and may be made to cancel for the SO(8) superstring.[6] It is thus interesting to demonstrate explicitly that these dilaton amplitudes cancel out in SO(32) superstring theory and examine what happens in the next order, since computation of dilaton tadpoles at the one-loop level is much easier than that of general amplitudes at the two-loop level. In this note we show that the dilaton tadpoles indeed vanish at tree as well as one-loop levels for the SO(32) type-I superstring *without ambiguity* after summation over spin structures.[7] This is a strong suggestion that the theory is free from divergences up to two loops.

There has been a "nonrenormalization" theorem[8] which claims that amplitudes with three or fewer external states vanish to all orders in superstring perturbation theory. These arguments, however, assume unbroken spacetime supersymmetry and mainly focus on closed superstrings, although it is plausible that they can be extended to open superstrings. In view of the fact that open superstring [except SO(32)] theory was found to suffer from gauge, gravitational, and most probably supersymmetry anomalies by explicit computation,[1] it is desirable to check the cancellation directly without assuming supersymmetry. My results suggest that the nonrenormalization theorem is probably valid also in this theory.

I use superfield[9,10] for the fermionic strings. Thus the coordinate superfield is defined by

$$V^\mu(z,\theta)=X^\mu+\theta\bar\lambda^\mu+\bar\theta\lambda^\mu+\theta\bar\theta F^\mu \quad (\mu=1,\ldots,10), \tag{1}$$

where z and θ are one-dimensional complex (Grassmann) numbers. With covariant derivatives $D=\partial_\theta+\theta\partial_z$, $\bar D=\partial_{\bar\theta}+\bar\theta\partial_{\bar z}$, the Lagrangean is written as

$$\begin{aligned} L&=\tfrac{1}{2}\int d^2\theta DV\cdot\bar DV \\ &=\tfrac{1}{2}(\partial_{\bar z}X\cdot\partial_z X-\lambda\cdot\partial_z\lambda-\bar\lambda\cdot\partial_{\bar z}\bar\lambda+F^2). \end{aligned}\tag{2}$$

It follows that the propagator on the sphere is given by[9,10]

$$\langle V^\mu(z,\theta)V^\nu(z',\theta')\rangle = -(\delta^{\mu\nu}/\pi)\ln|z-z'-\theta\theta'|^2. \tag{3}$$

There is only one spin structure (boundary condition) since the sphere has no uncontractible loop. Equation (3) corresponds to periodic fermion fields on the sphere realized by fields with Neveu-Schwarz boundary conditions in cylindrical coordinates.[10]

The path integral for type-I superstring theory contains a sum over all world-sheet topology.[11,12] By use of the dilaton vertex

$$\kappa e^{\mu\nu}\int d^2z\, d^2\theta DV^\mu\bar DV^\nu e^{ip\cdot V}, \tag{4}$$

with

$$e^{\mu\nu}=\delta^{\mu\nu}-p^\mu\bar p^\nu-p^\nu\bar p^\mu,\quad p\cdot\bar p=1,\quad p^2=\bar p^2=0, \tag{5}$$

it is easy to derive the dilaton tadpole amplitude

$$A(p)=(2\pi)^{10}\delta(p)\lambda^{-\chi}\kappa\frac{1}{|V_{\mathrm{MCG}}|\tilde{d}}\sum_{\mathrm{spin}}\left(\frac{\det P_1^\dagger P_1}{\det P_{1/2}^\dagger P_{1/2}}\right)^{1/2}\left(\frac{\det' \boldsymbol{D}}{\det\nabla^2}\right)^5 e^{\mu\nu}\int d^2z\,d^2\theta \lim_{\substack{z'\to z\\ \theta'\to\theta}} D\bar{D}'\langle V^\mu(z,\theta)V^\nu(z',\theta')\rangle, \quad (6)$$

where the sum over surfaces and integration over (super-)Teichmüller parameters are suppressed, ∇^2 and $\boldsymbol{D}$ are Laplacian and Dirac operators, respectively, and P_1 is a differential operator that maps vector into second rank tensor with $P_{1/2}$ its supersymmetric analogue. Equation (6) contains the sum over spin structures (if any), and is divided by the order $\tilde{d}$ of the diffeomorphism group and the volume $|V_{\mathrm{MCG}}|$ of the mapping class group, with the power of the coupling constant λ given by the Euler number χ of the world sheet. Note that among various possible contractions, only the one in Eq. (6) remains because of $e\cdot p=p^2=0$, and (6) is valid to the order of our interest. Thus our task is reduced to computing determinants and the propagator (Neumann function, on appropriate Riemann surfaces.

The lowest surfaces on which the dilaton tadpole in the type-I superstring theory appears are the disk (D_2) and projective plane (RP_2). The Neumann functions for these spaces are obtained from those on their "doubles" (closed oriented surfaces) by symmetrization with respect to the corresponding involutions.[13] Our basic idea is to use a superspace generalization of this technique.[9] In the present case, the double is a sphere and the apropriate involutions are

$$z\to k/\bar{z},\quad \theta\to \pm(-k)^{1/2}\bar{\theta}/\bar{z}, \quad (7)$$

where $k=+1$ (-1) corresponds to D_2 (RP_2). The involution for θ is fixed by the requirement that (7) preserve supersymmetry, i.e., $Q\to\bar{Q}$ up to an overall factor. Note the sign ambiguity in (7), corresponding to the ambiguity in defining the relative sign of $\bar{\lambda}$ to λ. No physical result will depend on this sign. From (3) one thus has

$$\langle V^\mu(z,\theta)V^\nu(z',\theta')\rangle = -(\delta^{\mu\nu}/\pi)\{\ln|z-z'-\theta\theta'|^2+\ln|1-k\bar{z}z'\pm(-k)^{1/2}\bar{\theta}\theta'|^2\}. \quad (8)$$

There is no (different) choice of spin structure on D_2 and RP_2.[12] This is to be expected because there is only one spin structure on their double (sphere). Since the scalar superfield (1) remains same under (7), one gets the condition on λ and $\bar{\lambda}$:

$$\lambda(k/\bar{z})=\pm(-k)^{1/2}z\bar{\lambda}(z),\quad \bar{\lambda}(k/\bar{z})=\pm\bar{z}\lambda(\bar{z})/(-k)^{1/2}. \quad (9)$$

This is in accordance with the condition for a fermion to be well defined on D_2 and RP_2 (i.e., under the map $z\to k/\bar{z}$); one can in fact give mode expansions for λ and $\bar{\lambda}$ satisfying (8) and (9) in terms of Neveu-Schwarz mode operators.

The bosonic determinants for D_2 and RP_2 were evaluated in Ref. 4 and their ratio is given by $RP_2/D_2=2^5$ in our case. Therefore we only have to compute (the ratio of) the fermionic determinants. Now spinors on a sphere with angular momentum (j,m) are given by[14]

$$\psi_1=\begin{pmatrix}[(j+m)/2j]^{1/2}Y_{j-1/2,m-1/2}\\ [(j-m)/2j]^{1/2}Y_{j-1/2,m+1/2}\end{pmatrix},\quad \psi_2=\begin{pmatrix}-[(j-m+1)/(2j+2)]^{1/2}Y_{j+1/2,m-1/2}\\ [(j+m+1)/(2j+2)]^{1/2}Y_{j+1/2,m+1/2}\end{pmatrix}. \quad (10)$$

Under (7), they transform as ($i=1,2$)

$$\psi_i(\pi-\theta)=(-1)^{j-m+i-1}\sigma_3\psi_i(\theta),\quad \psi_i(\pi+\theta)=(-1)^{j+1/2+i}\psi_i(\theta).$$

As a consequence, we can expand spinors as

$$\psi=\sqrt{2}\sum_j\sum_{m=-j}^{j}(c_{jm}^{(1)}\psi_1+c_{jm}^{(2)}\psi_2),$$

where $c_{jm}^{(1)}=0$ for even (or odd) $j-m+i$ for D_2, and $c_{jm}^{(i)}=0$ for even (or odd) $j-\frac{1}{2}+i$ for RP_2 ($i=1,2$). These coefficients are determined from the requirement that the scalars $\psi^\dagger\psi$ take the same value at the involution points. For both D_2 and RP_2, there is always one spinor (but not both ψ_1,ψ_2) for each (j,m). Since the eigenvalues of $\boldsymbol{D}$ depend only on j and $\det(P_{1/2}^\dagger P_{1/2})$ is essentially the same as $\det'\boldsymbol{D}$,[11] this implies that fermionic determinants are equal for D_2 and RP_2.

Substituting these results into (6), one finds

$$A_D(p)=(2\pi)^{10}\delta(p)\lambda^{-1}\kappa'\frac{1}{|V_{\mathrm{MCG}}|\tilde{d}}\int d^2z\frac{k}{(1-k|z|^2)^2}\times\begin{cases}2^{-5}\text{ for }D_2,\\ 1\text{ for }RP_2,\end{cases} \quad (11)$$

which is similar to the bosonic case.[4] Note that the sign ambiguity disappears here. It is now necessary to factor out the volume of mapping class from the integral and evaluate the ratio of the integral and $|V_{\mathrm{MCG}}|\tilde{d}$. Proceeding as in Ref. 4, one finally finds

$$A_D(p)=(2\pi)^{10}\delta(p)\lambda^{-1}\kappa'\times\begin{cases}2^{-5}\text{ for }D_2,\\ -1\text{ for }RP_2.\end{cases} \tag{12}$$

Since the amplitude on D_2 gets a group factor N for SO(N), this implies that the total amplitude vanishes for SO(2^5), as expected.[1,2]

I now wish to extend the analysis to one-loop level. At this level, there are four surfaces: torus (T_2), cylinder (C_2), Möbius strip (M_2), and Klein bottle (K_2). The latter three may be constructed from their double T_2 by imposition of suitable symmetries under involution.[13] The argument is thus a generalization of the proof of vanishing tadpoles on T_2 for closed superstring,[12] combined with these involutions. So it is instructive to review this case[15] first.

A torus can be represented as the complex plane with periodicity $z\approx z+1\approx z+\tau$ where τ is a complex number called Teichmüller parameter. There are four possible spin structures with the boundary conditions

$$\bar{\lambda}^\mu(z+n\tau+m)=(-1)^{n(1-\alpha)+m(1-\beta)}\bar{\lambda}^\mu(z), \tag{13}$$

with α, $\beta=0,1$, and similarly for λ^μ with independent set (α',β'). It is known that the amplitude in the sector (1,1) vanishes because of fermion zero modes,[12,15] and we can neglect this case. The fermionic determinant in the (α,β) sector is given by

$$(\det P^\dagger_{1/2}P_{1/2})^{-1/2}(\det{}'\boldsymbol{D})^5=\eta_{(\alpha\beta)}\{\theta[^\alpha_\beta](0)/(\theta'[^1_1](0))^{1/3}\}^4, \tag{14}$$

where $\theta[^\alpha_\beta](z)$ is the Jacobi θ function with the characteristic $(\alpha\beta)$, θ' is the derivative of θ with respect to z, and $\eta_{(\alpha\beta)}$ is a phase. The modular invariance of the theory tells us that

$$\eta_{(\alpha\beta)}=(-1)^{\alpha+\beta}, \tag{15}$$

up to an overall phase.[7] Now the Neumann function for T_2 is

$$\langle V^\mu V^\nu\rangle T\sim\delta^{\mu\nu}\left[\frac{1}{\pi}\ln\left|\frac{\theta[^1_1](z-z')}{\theta'[^1_1](0)}\right|^2-\frac{2[\operatorname{Im}(z-z')]^2}{\operatorname{Im}\tau}+\frac{\theta\theta'}{\pi}\frac{\theta[^\alpha_\beta](z-z')}{\theta[^1_1](z-z')}\frac{\theta'[^1_1](0)}{\theta[^\alpha_\beta](0)}\right.$$
$$\left.+\frac{\bar{\theta}\bar{\theta}'}{\pi}\frac{\theta[^{\alpha'}_{\beta'}](\bar{z}-\bar{z}')}{\theta[^1_1](\bar{z}-\bar{z}')}\frac{\theta'[^1_1](0)}{\theta[^{\alpha'}_{\beta'}](0)}\right]. \tag{16}$$

The fermionic propagator is determined by the requirement that it be meromorphic with a pole at $z=z'$ with residue $1/\pi$ and satisfy suitable boundary conditions. Substituting (14)–(16) into (6), one finds that only the bosonic (first two) terms in (16) contribute to (6). Thus the spin-structure dependence of (6) is solely given by (14), and hence (6) vanishes after summation over spin structures because of the identity

$$\sum_{\alpha,\beta}(-1)^{\alpha+\beta}\theta[^\alpha_\beta](0)^4=0. \tag{17}$$

The arguments for other amplitudes are basically the same combined with the supersymmetric involutions applied to T_2. In the bosonic case, the Neumann function for C_2 can be obtained by the involution $z\to-\bar{z}$ from T_2 with the Teichmüller parameter $\tau=it$ (pure imaginary).[13] In our case we should add the involution $\theta\to\pm i\bar{\theta}$. Thus

$$\langle V^\mu V^\nu\rangle_C=\langle V^\mu V^\nu\rangle_{T|\tau=it}+\langle V^\mu V^\nu\rangle_T\big|_{\tau=it,z\to-\bar{z},\theta\to\pm i\bar{\theta}}, \tag{18}$$

and the fermionic determinant is obtained from (14) by our putting $\tau=it$.[12] One again finds that only bosonic terms contribute to (6) and immediately gets the vanishing result for dilaton tadpole on C_2 after summation over spin structures. Note, however, that in this case it is not the modular invariance but another argument involving proper spin and statistics that determines the phase (15), which corresponds to Gliozzi-Scherk-Olive projection.[12]

It would be now fairly clear that the same argument applies to other cases M_2 and K_2 once suitable involutions are given. So I only give their involutions and τ: for M_2,

$$\tau=\tfrac{1}{2}(1+it)\text{ (no involution necessary);}$$

for K_2,

$$\tau=it,\quad z\to-\bar{z}+\tfrac{1}{2}(1+it),\quad \theta\to\pm i\bar{\theta}.$$

In this way one finds that these amplitudes separately vanish after summation over spin structures. Since this summation has the effect of projecting to supersymmetric states,[7,12] these results are consequences of unbroken supersymmetry.

The above results are in accordance with the fact that the dilaton vertex at zero momentum is just the string action and so the dilaton tadpole is obtained by differentiation, with respect to the string tension, of the partition function, which vanishes at one loop[12] after the spin structure is summed. [In this connection, it is a little puzzling that the dilaton amplitude vanishes but the vacuum energy seems to be nonzero for the SO(8192) bosonic string.[4]]

Thus supersymmetry is clearly very important in the cancellation at one-loop level, but it appears that SO(32) is necessary only in the lowest order. Of course, supersymmetry is anomalous[1] except for SO(32) and obviously there is a deep connection between supersymmetry, finiteness, and SO(32) symmetry. I hope that my results shed some light on this problem. Especially I wish to emphasize that my computation does not have an ambiguity in regularization[6] and clearly indicates that infinities will cancel for SO(32). It would be interesting to see how much these discussions can be carried over to higher loops. Finally, I remark that the method in this paper can also be applied to heterotic strings.

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Note added.—After completing this work, I received a revised version of the last paper in Ref. 9 in which a similar result for D_2 and RP_2 is also obtained.

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