# Slow-Motion Scattering and Coalescence of Maximally Charged Black Holes

Robert C. Ferrell
*Department of Physics, University of California, Santa Barbara, Santa Barbara, California 93106*

and

Douglas M. Eardley
*Institute for Theoretical Physics, University of California, Santa Barbara, Santa Barbara, California 93106*
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We study systems consisting of several maximally charged, nonrotating black holes ("Reissner-Nordstrom" black holes) interacting with one another. We present an effective action for the system in the slow-motion, fully strong-field regime. We give an exact calculation of black-hole–black-hole scattering and coalescence in the slow-motion (but strong-field) limit.

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The two-body problem for black holes can be solved in at least one known case, the limiting case of two widely separated, slowly moving black holes. This is the slow-motion, weak-field limit of gravity, and the problem reduces to the Kepler problem—the two-body problem of Newtonian gravity. In this Letter we will solve the two-body problem for black holes in another limiting case, that of two maximally charged, nonrotating black holes that move slowly, but that are close to one another—i.e., a system of two Reissner-Nordstrom[1] black holes with charge $Q$ and mass $M$, related by $Q=G^{1/2}M$, with speeds $v \ll c$ and separations $r \simeq GM/c^2$. Even though restricted to the slow-motion limit, this is a problem in the full strong-field regime of gravity coupled to electromagnetism, and it is perhaps surprising that the problem can be solved analytically. Much work on the *uncharged* $n$-body problem is general relativity has shown it to be a difficult problem, even for $n=2$.[2] Here we show that the problem becomes much simpler for suitably charged bodies.

For maximally charged black holes, the motion remains slow despite the strong fields, because (roughly speaking) the electrostatic repulsion cancels the gravitostatic attraction, and the black holes move under the influence of magnetic and gravitomagnetic (or "frame-dragging") forces, which remain small as long as the velocities remain small.[3,4] This problem is almost certainly irrelevant for astrophysics, since such highly charged black holes would immediately neutralize themselves by plasma processes. However, similar problems involving rotating but uncharged black holes, or black holes which have swallowed magnetic monopoles and become magnetically charged, might also be of astrophysical interest.

The slow-motion, strong-field problem has recently been solved for magnetic monopoles in gauge theories[5-7] and in five-dimensional Kaluza-Klein theories,[8] and for solitons in the $CP^1$ model,[9] on the basis of a method proposed by Manton.[5] In this method the kinetic energy is viewed as a metric on configuration space and the trajectories of the dynamical system become the geodesics of this metric. Gibbons and Ruback[4] recently discussed the slow-motion, strong-field problem for maximal Reissner-Nordstrom black holes, and they suggested that Manton's method would likewise apply. We will show here that it does.

A static system of $n$ maximally charged nonrotating black holes has four-metric and four-potential[3,10-12]

$$ds^2=-\psi^{-2}dt^2+\psi^2 d\mathbf{x}^2, \tag{1a}$$

$$A=-(1-\psi^{-1})dt, \tag{1b}$$

where

$$\psi=1+\textstyle\sum_a m_a/r_a; \tag{2}$$

here the $a,b,c\ldots=(1,2\ldots n)$ are labels on the $n$ black holes, each of mass $m_a$, charge $q_a=m_a$, and position $\mathbf{x}_a$; and $\mathbf{r}_a=\mathbf{x}-\mathbf{x}_a$, $r_a=|\mathbf{r}_a|=|\mathbf{x}-\mathbf{x}_a|$. Units are such that $G=1=c$. In this coordinate system, the event horizons of the black holes lie at the points $\mathbf{x}=\mathbf{x}_a$.

We calculate the motions of the black holes perturbatively, in a slow-motion expansion. A difficulty arises: The fields are singular at the $\mathbf{x}_a$ and infinities occur in the calculations. We choose to regularize the problem by spreading out the sources a little: We replace the pointlike black holes with a smooth distribution of charged dust with density $\rho$ (pressureless with charge-to-mass ratio unity) and velocity $\mathbf{v}$. Then $\psi$ is also smooth and given by

$$\nabla^2\psi=-4\pi\psi^3\rho, \tag{3}$$

with boundary condition $\psi=1$ at infinity, where $\nabla^2$ is the Laplacian operator in the $\{\mathbf{x}\}$ regarded as coordinates on flat $\mathbf{R}^3$. As the final stage of the calculation, $\rho$ will be allowed to tend to a sum of point masses,

$$\psi^3\rho \to \textstyle\sum_a m_a\delta^{(3)}(\mathbf{x}-\mathbf{x}_a). \tag{4}$$

At this stage, all the infinities will, miraculously enough, cancel out, to leave a finite answer—*regularization* is necessary, but not *renormalization*. A similar attempt to treat the equations of motion of uncharged black holes would fail; the maximally charged case is actually much easier. Mathematically, it seems that a point particle with $q=m$ can be represented by a distribution, whereas an uncharged particle cannot.

The calculation is now straightforward; details will be published elsewhere. In outline: Allow the positions $\mathbf{x}_a(t)$ to be slowly varying functions of time $t$, and define three-velocities $\mathbf{v}_a \equiv d\mathbf{x}_a/dt$, which will be assumed small and treated as being of first order $O(v)$ in the slow-motion expansion. Allow the metric and potential fields to acquire perturbations also. The $x_a(t)$ will be the dynamical variables in an effective action $S_{\text{eff}}$ to be derived from the exact action $S$ in the $O(v^2)$ approximation. Since we want first-order equations of motion, we need to expand the exact action $S$ through terms of $O(v^2)$ around the exact, static solution (1) and (3). To $O(v^2)$ in the perturbed Einstein-Maxwell equations, the field perturbations are entirely determined in terms of the matter perturbations by constraint equations (i.e., no radiation appears). Solve the constraints to express the fields in terms of the matter variables $\{\rho,\mathbf{v}\}$. Substitute these solutions into $S$ to obtain the effective action $S_{\text{eff}}$. Take the black-hole limit (4) to obtain an $S_{\text{eff}}$ which is a functional only of the $x_a(t)$. Finally, vary $S_{\text{eff}}$ with respect to the $x_a(t)$ to derive first-order equations of motion for the black holes.

The exact action of the system is

$$S=S_{\text{grav}}+S_{\text{em}}+S_{\text{current}}+S_{\text{matter}}$$
$$=\frac{1}{16\pi}\int d^4x\sqrt{-g}R+(\text{bdy})-\frac{1}{16\pi}\int d^4x\sqrt{-g}F^2+\int d^4x\sqrt{-g}A_\mu\rho u^\mu-\int d^4x\sqrt{-g}\rho. \tag{5}$$

Here $u^\mu$ is the matter four-velocity, $A=A_\mu dx^\mu$, and $F$ is the field strength. The boundary term of gravity is (bdy) which we need not discuss further here. The functions to be varied are the fields $g_{\mu\nu}$ and $A_\mu$, and the world lines $x^\mu(s)$ of the matter trajectories; $\rho$ is not varied freely, but rather is adjusted to keep the matter conserved under the other variations. The static solution is an extremum of the exact action, so that terms linear in second-order field perturbations vanish exactly in this expansion, and only terms quadratic in first-order field perturbations survive through $O(v^2)$. Furthermore, the first-order perturbations in quantities which are even under time reversal vanish; these include first-order perturbations in $g_{tt}$ and $g_{ij}$, and in $A_t$. Thus, a general enough form of the perturbed metric and potential is

$$dt^2=-\psi^{-2}dt^2+2\mathbf{N}\cdot d\mathbf{x}\,dt+\psi^2 d\mathbf{x}^2, \tag{6a}$$
$$A=-(1-\psi^{-1})dt+\mathbf{A}\cdot d\mathbf{x}, \tag{6b}$$

where $\psi$ is defined by (2), and the only field perturbations that contribute are the first-order quantities $\mathbf{N}$ and $\mathbf{A}$.

In fact, we shall use these quantities in combinations $\mathbf{P}$ and $\mathbf{Q}$ which transform simply under gauge transformations:

$$\mathbf{P}\equiv\mathbf{A}-\psi\mathbf{N}, \tag{7a}$$
$$\mathbf{Q}\equiv\psi^2\mathbf{N}. \tag{7b}$$

In terms of the first-order quantities $\mathbf{P}$ and $\mathbf{Q}$, the first-order field equations are

$$\nabla\times\left[\frac{\nabla\times\mathbf{P}}{\psi^2}-\frac{\nabla\times\mathbf{Q}}{\psi^3}\right]=\nabla\times(\nabla\times\mathbf{K}), \tag{8a}$$

$$\nabla\times\left[\frac{\nabla\times\mathbf{P}}{\psi^3}-\frac{3\nabla\times\mathbf{Q}}{4\psi^4}\right]=0, \tag{8b}$$

where the vector operations are with respect to the $\mathbf{x}$ regarded as flat coordinates on $\mathbf{R}^3$, and where $\mathbf{K}\equiv-4\pi\times\nabla^2(\rho\psi^3\mathbf{v})$. In the black-hole limit, $\mathbf{K}\to\sum_a m_a\mathbf{v}_a/r_a$. Equations (8) are linear combinations of Ampère's law, and of the first-order supermomentum constraint in the Arnowitt-Deser-Misner[1,13] Hamiltonian formulation of general relativity. The Gauss's-law constraint and the zeroth-order Hamiltonian constraint are solved by (1) and (2); these latter constraints are time even and therefore are trivial in first order.

The solutions of (8) are

$$\nabla\times\mathbf{P}=-3\psi^2\nabla\times\mathbf{K}-2\nabla\alpha-3\psi^2\nabla\nu, \tag{9a}$$
$$\nabla\times\mathbf{Q}=-4\psi^3\nabla\times\mathbf{K}-4\psi\nabla\alpha+4\alpha\nabla\psi-4\psi^3\nabla\nu; \tag{9b}$$

here the first-order scalar fields $\alpha$ and $\nu$ are functions of integration, which can be determined by taking the divergences of (9). In the black-hole limit (4), the contribution of the scalar fields $\alpha$ and $\nu$ to $S_{\text{eff}}$ vanishes.

Substituting (9) into $S$ and keeping terms up through $O(v^2)$, we take the black-hole limit (4) and find that $S$ approaches a finite limit $S_{\text{eff}}$. The action $S_{\text{eff}}$ governs the firs-order motion of $n$ maximally charged nonrotating black holes in the slow-motion limit:

$$S_{\text{eff}}=\int dt\,(L_{\text{free}}+L_{\text{int}}),$$

where

$$L_{\text{free}}=-\sum_a m_a+\sum_a \tfrac{1}{2}m_a v_a^2, \tag{10a}$$
$$L_{\text{int}}=\frac{3}{8\pi}\int d^3x\left[1+2\sum_c\frac{m_c}{r_c}+\sum_{cd}\frac{m_c m_d}{r_c r_d}\right]\sum_{ab}\frac{m_a m_b}{r_a^3 r_b^3}\left[\tfrac{1}{2}(\mathbf{r}_a\cdot\mathbf{r}_b)\,|\mathbf{v}_a-\mathbf{v}_b|^2-(\mathbf{r}_a\times\mathbf{r}_b)\cdot(\mathbf{v}_a\times\mathbf{v}_b)\right]. \tag{10b}$$

This action has the expected properties. It is the sum of a free part and an interaction part that is invariant under Galilean boosts $\mathbf{v}_a \rightarrow \mathbf{v}_a + \mathbf{u}$ where $\mathbf{u}=$(a constant three-velocity). It reproduces previously known results[4] in the Newtonian limit and in the test-particle limit. Remarkably, $L_{\text{int}}$ is a polynomial interaction among the black holes, consisting precisely of quadratic, cubic, and quartic terms in the masses $m_a$. Therefore, the black holes experience only two-body, three-body, and four-body interactions. Variation of the $\mathbf{x}_a(t)$ in $S_{\text{eff}}$ gives the first-order equations of motion for $n$ black holes; the full space-time metric and potential can be reconstructed from Eq. (2), (6), (7), and (9).

Turn now to the two-body problem. The Lagrangean $L_{\text{free}}+L_{\text{int}}$ becomes

$$L_{\text{two body}} = -M + \tfrac{1}{2} M\mathbf{V}^2 + \tfrac{1}{2}\mu \mathbf{v}^2\{[1+M/r]^3 - 2\mu M^2/r^3\}, \tag{11}$$

where, as usual, $M=m_1+m_2$, $\mu = m_1 m_2/M$, $\mathbf{V}=(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/M$, and $r=|\mathbf{x}_1 = \mathbf{x}_2|$. This system is completely integrable, and allows exact calculations of black-hole scattering and coalescence in the slow-motion limit.

Let the two black holes approach each other from infinity with asymptotic speed $v_\infty \ll 1$ and impact parameter $b$. As long as the slow-motion approximation is valid (see below), the speed $v_\infty$ scales out of the problem, and the outcome depends only on $b/M$. We find that there is a critical value $b_{\text{coal}}$ of $b$, outside of which the black holes scatter back to infinity, and inside of which they coalesce. In general, $b_{\text{coal}}$ is found by solving the polynomial equation

$$0 = -\tfrac{4}{3} b^6 + 9M^2 b^4 - 36\mu M^3 b^2 - 36\mu^2 M^4.$$

In particular, for equally massive black holes, $m_1 = m_2$,

$$b_{\text{coal}} = [3+\tfrac{3}{2}\sqrt{3}]^{1/2} M \approx 2.3660M,$$

to be compared with the known value for black hole and test particle,

$$b_{\text{coal}} = (\tfrac{27}{4})^{1/2} M \approx 2.5981M.$$

In the two-body problem, the slow-motion approximation remains valid for $r \gg v_\infty^2 M$. For scattering cases, it remains valid throughout, and little energy is radiated in electromagnetic or gravitational radiation, $\Delta E_{\text{rad}} \simeq v_\infty^5 M$. The two black holes return to infinity as $t \rightarrow \infty$.

For coalescence cases, the two black holes approach each other as $t\rightarrow\infty$ according to $r \propto t^{-2}$. The slow-motion approximation eventually breaks down as the two black holes approach to $r \simeq v_\infty^2 M$. For coalescences with $b < \mu^{-1/2} M^{3/2}$ the system has a small enough angular momentum to form a nearly maximal Kerr-Newman black hole. Simple estimates suggest that a new black hole has formed before the slow-motion approximation breaks down; thus, in this case we expect there to be little radiation, $\Delta E\text{rad} \simeq v_\infty^5 M$. In contrast, for coalescences with $b > \mu^{-1/2} M^{3/2}$, the system has too much angular momentum to form a Kerr-Newman black hole. We speculate that in this case the system emits a substantial amount of radiation energy $\Delta E_{\text{rad}} \simeq v_\infty^2 M$ after the slow-motion approximation breaks down at $r \simeq v_\infty^2 M$, and then it settles down to a black hole. Technically, formation of a naked singularity or escape are also possibilities. These latter ($b > \mu^{-1/2} M^{3/2}$, "radiation-loud") interactions occur only for mass ratios $0.21002 \lesssim m_1/M \lesssim 0.78998$.

According to Manton's[5] scheme, the coefficients of the quadratic form over the $\mathbf{v}_a$ in $L$ are to be interpreted as a metric form on configuration space $\mathbf{R}^{3n}$. For the two-body problem (11), this metric is

$$ds^2_{\text{Manton}} = M\,d\mathbf{R}^2 + \mu[(1+M/r)^3 - 2\mu M^2/r^3]d\mathbf{r}^2; \tag{12}$$

it is Riemannian and complete. The geometry of configuration space $\mathbf{R}^6$ under this metric is a flat $\mathbf{R}^3$ in the c.m. coordinates $\{\mathbf{R}\}$ times a curved $\mathbf{R}^3$ in the $\{\mathbf{r}\} \equiv \{\mathbf{x}_1 - \mathbf{x}_2\}$. Because angular momentum is conserved, the motion is restricted to a two-dimensional surface embedded in the curved $\mathbf{R}^3$. This surface is plotted in Fig. 1. The surface is asymptotically flat as $r\rightarrow\infty$, and asymptotically an infinite cone as $r \rightarrow 0$, the conical deficit angle being $\pi$. In this diagram, a point on the surface represents a three-geometry, by Eq. (2). A geodesic on this surface represents an evolution of the black holes—and consequently a space-time, through Eqs. (9). The evolutions which result in coalescence are represent-

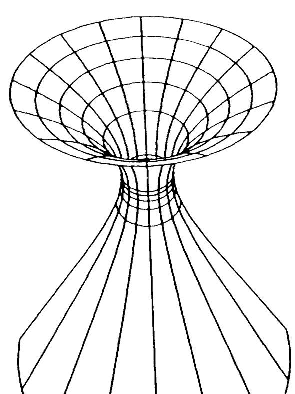

FIG. 1. Manton geometry for the case $m_1 = m_2$, restricted to a two-dimensional surface. A geodesic on this surface represents the evolution of the two black-hole system. Circles represent lines of constant separation of the black holes, $r=|\mathbf{x}_1 - \mathbf{x}_2|$. The throat is at $r = M\{[(b_{\text{coal}}/M)^2/3]^{1/2} - 1\} \simeq 0.366M$. Circles above the throat are separated by $\Delta r = 1.46M$, below the throat by $\Delta r = 0.0731M$. The point $r=0$ lies infinitely far down the cone.

ed by those geodesics which start at $r=\infty$ and pass through the throat onto the cone.

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