Self-Organization of Electrostatic Turbulence in a Cylindrical Plasma

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On the basis of theory and computer simulations we show that electrostatic turbulence in a cylindrical plasma with magnetic shear and curvature self-organizes to form a macroscopic potential ϕ which depends only on the radial coordinate r and is given by $\phi(r) \simeq J_0(pr) + C_1 r^2 + C_2$, where C_1 and C_2 are functions of a constant p. A unique feature of the potential is the existence of a coaxial $\phi(r_0)=0$ surface at $r_0 \simeq 0.7a$, where a is the radius of the cylinder. This surface is found to be fairly rigid and is considered to inhibit radial particle transport.

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In the presence of appropriate constraints, plasma turbulence is known to self-organize to form semicoherent macroscopic structures[1-3] which play crucial roles in the equilibria and transports. In this Letter we present the first evidence of self-organization of electrostatic turbulence in a cylindrical plasma based on three-dimensional simulations and a theory. The plasma turbulence is excited by a combination of the resistive drift-wave instability and the resistive interchange instability[4] in the presence of an axisymmetric magnetic field with a curvature and shear. The conservation of potential enstrophy induces condensation of the turbulence energy into the zero axial and zero azimuthal eigenmode to form an axisymmetric potential surface $\phi(r)$. The conservation of angular momentum uniquely determines the structure of the potential $\phi(r)$ in the form

$$\phi(r) = (1 - 4/p^2) J_0(pr)/J_0(p) - r^2 + 4/p^2,$$

where $p = 3.82$, r is normalized to unity at the radius of the cylinder, and J_0 is the Bessel function.

The model equations we use to describe the electrostatic turbulence are the equation of vorticity,

$$(\rho_s^2/a^2) d(\nabla_\perp^2 \phi)/dt = (\nabla \ln n \times \nabla \Omega)\cdot \hat{\mathbf{z}} + (\omega_{ce}/\nu_{ei})(a/R)^2 \nabla_\parallel^2 (\ln n - \phi) + (\mu/\omega_{ci} a^2)\nabla_\perp^4 \phi, \tag{1}$$

and the equation of continuity,

$$d \ln n/dt = (\nabla \ln n \times \nabla \Omega)\cdot \hat{\mathbf{z}} + (\omega_{ce}/\nu_{ei})(a/R)^2 \nabla_\parallel^2 (\ln n - \phi). \tag{2}$$

Here the parallel current $J_\parallel$ is eliminated by the use of a generalized Ohm's law with the assumed isothermal electron pressure gradient. The contribution of the ion parallel current is ignored. $\phi\ (=e\phi/T_e)$ is the normalized electrostatic potential, n is the plasma number density, a is the radius of the cylinder, ω_{ce} is the electron cyclotron frequency, ν_{ei} is the electron-ion collision rate, and ρ_s $[=(T_e/m_i)^{1/2}/\omega_{ci}]$ is the ion Larmor radius at the electron temperature. In the derivation of Eq. (1) from the ion equation of motion, the gradient of mass density is ignored with respect to the gradient of ϕ. The perpendicular coordinate is normalized to the cylinder radius a, the parallel coordinate is normalized to the major radius R, and the time is normalized by $(\omega_{ci}\rho_s^2/a^2)^{-1}$. The convective derivative is given by

$$d/dt = \partial/\partial t + \mathbf{v}_E \cdot \nabla, \tag{3}$$

$$\mathbf{v}_E = -\nabla\phi \times \hat{\mathbf{z}}, \tag{4}$$

and the parallel derivative is given by

$$\nabla_\parallel = \partial/\partial z + (\nabla\psi_0 \times \hat{\mathbf{z}})\cdot\nabla. \tag{5}$$

The curvature term Ω and the flux function ψ_0 may be represented by the rotational transform $\iota(r)$ for a heliotron configuration,

$$\psi_0 = \int_0^r r\iota(r)\,dr \tag{6}$$

and

$$\Omega = (a^2/R^2)(N/l)[r^2\iota(r) + 2\int_0^r r\iota(r)\,dr], \tag{7}$$

where l is the pole number and N is the pitch number. By subtracting Eq. (2) from (1), one can construct the equation for the potential vorticity ζ,

$$d\zeta/dt=(\mu/\omega_{ci}a^2)\nabla_\perp^2\phi, \qquad (8)$$

where

$$\zeta=(\rho_s^2/a^2)\nabla_\perp^2\phi-\ln n. \qquad (9)$$

The potential enstrophy conservation results immediately from Eq. (8):

$$\frac{\partial}{\partial t}\frac{\zeta^2}{2}+\nabla\cdot\left[\mathbf{v}_E\frac{\zeta^2}{2}\right]=\frac{\mu}{\omega_{ci}a^2}\zeta\nabla_\perp^4\phi, \qquad (10)$$

where $U=\int(\zeta^2/2)dV$ is the potential enstrophy. The energy conservation is obtained by the multiplication of Eq. (1) by ϕ and (2) by $\ln n$, and subtraction of the result,

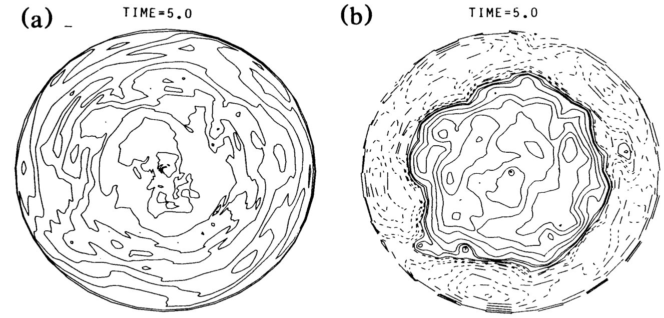

FIG. 1. (a) The density contour and (b) the potential contour from the three-dimensional computer simulation of electrostatic plasma turbulence in a cylindrical plasma with magnetic curvature and shear. In (b) the solid (dashed) lines are for the positive (negative) potential contours. Note the development of closed potential contours near the $\phi\simeq0$ surface.

$$\frac{\partial}{\partial t}\left[\frac{(\ln n)^2}{2}+\frac{\rho_s^2}{a^2}\frac{(\nabla_\perp\phi)^2}{2}\right]+\nabla\cdot\left[\frac{1}{2}\mathbf{v}_E\left[(\ln n)^2-\frac{\rho_s^2}{a^2}\phi\nabla_\perp^2\phi\right]-\frac{\rho_s^2}{a^2}\phi\frac{\partial}{\partial t}\nabla_\perp\phi\right]$$
$$=\nabla\left[\frac{(\ln n)^2}{2}-\phi\ln n\right]\cdot(\nabla\Omega\times\hat{\mathbf{z}})$$
$$+\frac{\omega_{ce}}{\nu_{ei}}\left[\frac{q}{R}\right]^2\{\nabla_\parallel[(\ln n-\phi)\nabla_\parallel(\ln n-\phi)]-[\nabla_\parallel(\ln n-\phi)]^2\}-\phi(\mu/\omega_{ci}a^2)\nabla_\perp^4\phi, \qquad (11)$$

where

$$E=\int\tfrac{1}{2}[(\ln n)^2+(\rho_s^2/a^2)(\nabla_\perp\phi)^2]dV$$

is the turbulent energy and the term on the order of $\nabla\cdot\mathbf{v}_E$ ($=\nabla\phi\cdot\nabla\Omega\times\hat{\mathbf{z}}$) is ignored. We note that under the assumption of a negligible $\nabla\cdot\mathbf{v}_E$, which is needed to construct the conservation laws, the curvature terms does not contribute either to the energy or to the enstrophy in spite of the fact that it contributes to the linear growth.

Linear instabilities which are obtained from Eqs. (1) and (2) are the resistive drift-wave instability, which is not localized at the mode rational surface as a consequence of the neglect of the parallel ion inertia, and the resistive interchange instability, which is localized at the mode rational surface.[4]

The computer simulation to study the nonlinear evolution of the instability was performed by expanding the potential $\phi(r,t)$ and the perturbed density $\tilde{n}(r,t)$ into the Fourier modes in the azimuthal and axial directions,

$$\phi=\textstyle\sum_{m,n}\phi_{m,n}\exp[i(m\theta-nz/R)],$$
$$\tilde{n}=\textstyle\sum_{m,n}\tilde{n}_{m,n}\exp[i(m\theta-nz/R)].$$

For computational purposes $\tilde{n}$ and the background density $n_0(r)$ are normalized to the peak density n_M, and $\nabla\ln n$ in Eqs. (1) and (2) is approximated by $\nabla[n_0(r)/n_M]+\nabla\tilde{n}$. The normalized density $n_0(r)/n_M$ is taken to be $n_0(r)/n_M=0.9\exp(-2r^2)+0.1$, and the rotational transform $\iota(r)=0.51+0.39r^2$. The radially increasing ι is chosen to simulate Stellarator-Heliotron fields. The mode number is selected within $|m|<20$ and $|n|<10$ which has rational surface between $\iota=0.5$ and 1.0. The total mode number is 111 including $m=0$, $n=0$ modes and chosen such that $1\le m/n\le2$. The radial coordinate is represented by 100 meshes. The time step Δt is selected to be $\simeq\Delta r/\frac{1}{5}c_s$, where Δr is the mesh size, and advanced by a predictor-corrector method. The boundary conditions are

$$\phi_{m,n}(0)=\tilde{n}_{m,n}(0)=V_{m,n}(0)$$
$$=\phi_{m,n}(1)=\tilde{n}_{m,n}(1)=V_{m,n}(1)=0$$

for $m\neq0$ and $n\neq0$, and $d\phi_{0,0}(0)/dr=dn_{0,0}^{(0)}/dr=0$ as well as $\phi_{0,0}(1)=\tilde{n}_{0,0}(1)=0$, where $V(=\nabla_\perp^2\phi)$ is the vorticity. Other parameters used in the computations are $\rho_s/a=\frac{1}{40}$, $a/R=\frac{1}{13}$, $\nu_{ei}/\omega_{ce}=1/7.5\times10^3$, $\mu/\omega_{ci}a^2=5\times10^{-4}$, and the diffusion coefficient $D_\perp/(\omega_{ci}a)^2$ introduced in Eq. (2) in order to saturate the instability is 5×10^{-4}. The initially small perturbation is given to $m=2,n=1$ and $m=3,n=2$ modes. Figures 1(a) and 1(b) show the perturbed density and potential contours at the saturation of the instability. The most conspicuous feature of the potential contours is the formation of the closed potential surface with $\phi(r)\simeq0$. This is a consequence of the generation of $m=0,n=0$ potential (which is an exact solution of the original set of equations) through the inverse cascade of the turbulent spectra. We note that the parity conservation in the conventional reduced MHD equations[5] does not allow the generation of a $\phi_{0,0}(r)$ mode as found here. We also note that the $\phi_{0,0}$ mode does not produce mode coupling back to produce $m\neq 0, n\neq 0$ and thus constitutes a condensed state. An additional characteristic feature of the potential $\phi_{0,0}$ is that it has a zero-potential surface within the plasma at $r\simeq 0.7$. The dashed and dash-dotted lines in Fig. 2 show the radial profile of $\phi_{0,0}(r)$ at two different time steps. We have run different parameter regimes with or without the curvature or fixed radial density profile; however, the conspicuous feature of the generation of $\phi_{0,0}(r)$ which has a zero surface at $r\simeq 0.7$ remained unchanged. Hence, we conclude that these features are a characteristic of the electrostatic turbulence in a cylindrical plasma which is pretty much independent of the detailed instability processes.

Let us now discuss the processes through which the coherent potential structure is formed. On the basis of the conservation of energy and enstrophy, the quasi two-dimensional electrostatic turbulence has been expected to form a self-organized state[6] in the form of a large-scale flow or a coherent potential profile. As usual we establish an appropriate variational principle to derive the self-organized structure. The existence of the $\phi(r)=0$ surface inside the plasma, as well as at the wall, means the existence of counter rotation of the plasma (a shear flow); hence, we introduce a constraint that the quantity $M\equiv\int(\zeta r^2/2)\,dV$ is conserved which can be derived straightforwardly from Eq. (8). The quantity M is equivalent to the angular momentum when an appropriate boundary condition is satisfied. As the variational form, we seek a solution which minimizes the total potential enstrophy U by keeping the total energy E and angular momentum M constant,

$$\delta U-\lambda_1\delta E-\lambda_2\delta M=0. \tag{12}$$

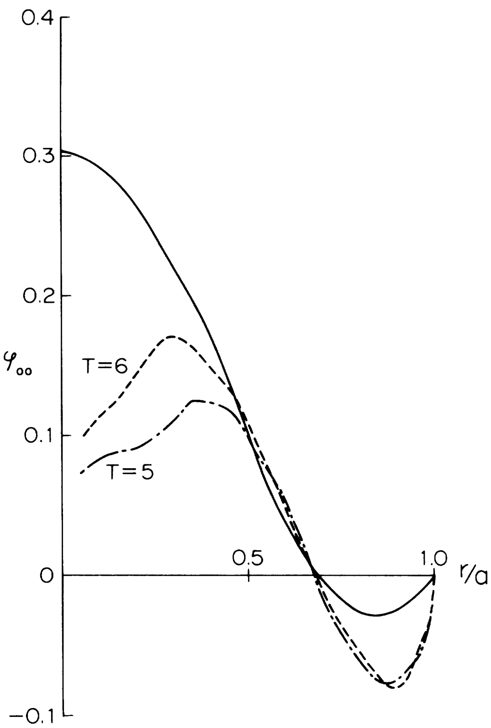

FIG. 2. Profiles of $\phi(r)$ for $m=0, n=0$ mode at two different time steps (dashed and dash-dotted lines) as compared with the predicted profile (solid line) based on the self-organization conjecture. The predicted curve is fitted at $r/a=0.5$.

If we take variations with respect to n and ϕ, Eq. (12) reduces to the following set of coupled equations:

$$(\lambda_1-1)\zeta-\lambda_1(\rho_s^2/a^2)\nabla_\perp^2\phi+\lambda_2 r^2/2=0, \tag{13}$$

$$\nabla_\perp^2\zeta+\lambda_1\nabla_\perp^2\phi-2\lambda_2=0. \tag{14}$$

The boundary conditions which are consistent with the simulations are $\phi(1)=\zeta(1)=0$. The solution of Eqs. (13) and (14) is readily obtained to give

$$\phi=-\frac{\lambda_2}{\lambda_1}\frac{a^2}{\rho_s^2}\frac{1}{2p^2}\left[\frac{1}{J_0(p)}\left(1-\frac{4}{p^2}\right)J_0(pr)-r^2+\frac{4}{p^2}\right] \tag{15}$$

and

$$\zeta=-\lambda_1\phi. \tag{16}$$

Here the eigenvalue p, which is related to λ_1 through $p^2=(a^2/\rho_s^2)(\lambda_1-1)$, is decided from the requirement $M=0$,

$$\frac{1}{2p}\frac{J_1(p)}{J_0(p)}\left(1-\frac{4}{p^2}\right)^2+\frac{3}{2p^2}-\frac{4}{p^2}-\frac{1}{12}=0, \tag{17}$$

and its smallest value is given by 3.82, which is close to the first zero of $J_1(p)$. The potential profile given by Eq. (15) is shown by the solid curve in Fig. 2. To compare with the simulation result, the theoretical profile obtained from Eq. (15) is fitted at $r=0.5$ Agreement with respect to the point of zero crossing and to the general profile is seen. The disagreement at $r\simeq 0$ and $r\simeq 1$ may be due to the fact that the turbulent energy in these regions is relatively low and, hence, it may take a longer period of time to reach the self-organized state. We also note that for any function $f(r)$, $\int f(r)\zeta^2 dV$ is conserved; thus the choice of $f=r^2/2$ is not mathematically unique.

A self-organization is often produced by condensation of the turbulent spectrum into the longest wavelength.

Consequently the potential profile obtained in Eq. (15) resembles the axial magnetic field of the Taylor's solution[2] for the reverse-field pinch. Taylor obtained the solution by minimizing the (magnetic) energy with the constraint of a constant helicity, while here the solution is obtained by minimization of the enstrophy rather than the energy, because enstrophy can be shown to dissipate more quickly than the energy for electrostatic turbulence.

In summary we have shown a self-organization of electrostatic turbulence of a cylindrical plasma with magnetic curvature and shear. The resultant axisymmetric potential profile is explained by means of the variational principle of minimization of the potential enstrophy. The self-organized axisymmetric potential contours show generation of azimuthal zonal flows; hence, they indicate inhibition of radial diffusion. In particular, near the $\phi=0$ surface, the radial particle flux is minimum since $\langle \phi n_1 \rangle \simeq 0$. This fact in turn produces a steep density gradient around this surface. The diffusion is likely to occur intermittently when a small cell moves across the closed stream lines.

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