Fast-Dynamo Action in Unsteady Flows and Maps in Three Dimensions

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Unsteady fast-dynamo action is obtained in a family of stretch-fold-shear maps applied to a spatially periodic magnetic field in three dimensions. Exponential growth of a mean field in the limit of vanishing diffusivity is demonstrated by a numerical method which alternates instantaneous deformations with molecular diffusion over a finite time interval. Analysis indicates that the dynamo is a coherent feature of the large scales, essentially independent of the cascade of structure to small scales.

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A current focus of kinematic dynamo theory is the search for *fast* dynamos, capable of generating exponentially growing magnetic fields in fluids of arbitrarily high electrical conductivity.[1,2] Although the qualitative arguments in favor of such dynamos are compelling, their explicit construction has proved elusive because of the complicated field geometry created in a highly conducting fluid by even the simplest movements. The difficulty is that most flows create regions of large field variation, so that magnetic field diffusion occurs on structures of ever-decreasing scale as the diffusivity tends to zero.[3]

The magnetic field in a highly conducting fluid behaves essentially like an ensemble of differential material line elements moving with the fluid. Spatially chaotic flows, in which material line elements stretch exponentially rapidly, are therefore natural candidates in which to seek fast dynamos. Special constructions on curved Riemannian manifolds indicate that continued field stretching can lead to a fast dynamo, and direct numerical simulations of field evolution in the chaotic *ABC* flows support this picture.[4,5] Strauss[6] has identified a complementary mechanism for fast-dynamo action in chaotic flows, utilizing an effective turbulent resistivity.

The role of chaotic flow with positive Liapunov exponent has been emphasized in fast-dynamo constructions[1,2,7,8] but line stretching *per se* is not sufficient for a fast dynamo. Even with exponential stretching, regions of oppositely oriented field can be squeezed together, with the resulting cancellation dissipating the field faster than it is being amplified. The SFS dynamo, described below, exhibits this effect for small values of the parameter α. A necessary condition for fast-dynamo action is that such destructive interference be minimized.

Recently, Soward[9] has found an example of fast-dynamo action by a steady, quasi two-dimensional velocity field, in which destructive interference effects are purposely minimized by the model. Although the flow is steady, the physical argument invokes an unsteady analogy. Conceptually, the field is first stretched by a two-dimensional flow—which also folds the field so that regions of opposite orientation are brought into proximity. The field is then sheared in the third direction, which (provided the field variation in this direction is chosen appropriately) rearranges the field so that the interference is predominantly constructive. We refer to this kind of process as the *stretch-fold-shear* (SFS) mechanism. In Soward's examples, it operates in thin magnetic diffusive layers at the cell boundaries of a periodic array of helical vortices.

Soward's work suggests the investigation of the SFS mechanism for simple, explicitly unsteady flows and maps in R^3. In the present note, we study the effect of a prescribed motion $\mathbf{u}(\mathbf{r},t)$ on a magnetic field $\mathbf{B}(\mathbf{r},t)$, which evolves according to[1]

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1}$$

where η is the magnetic diffusivity of the fluid. Our results indicate the existence of a large class of kinematic fast dynamos in which the motion consists of extremely rapid deformation followed by periods of rest during which (weak) diffusion acts. This separation allows the advective and diffusive processes to be treated separately, greatly simplifying the mathematical analysis.[10,11]

This paper examines two particular types of rapid motion. The first is a stretch in the x direction, a fold in y, and finally a shear in the z direction; this will be called the SFS *model* as its purpose is to exemplify the SFS mechanism in the least complicated way possible. This motion is discontinuous at the fold and therefore unrealistic, but has the advantage that it has extremely simple behavior. The second motion consists of rapid advection by a smooth velocity field, followed by a rotation of coordinates; this will be termed the *pulsed-flow model.* While such jerky motions are somewhat artificial, our models display the full range of behavior expected to occur in general three-dimensional dynamos. The theory does not require the machinery of multiscale boundary-layer theory, nor the existence of an effective chaotic diffusivity. The ease and transparency of the present constructions suggest that some dynamo action is a common feature of unsteady three-dimensional flows of electrically conducting fluids.

We consider a cube of side L described by the rescaled (dimensionless) coordinates $0 \leq x,y,z \leq 1$ in R^3. The

SFS map takes (x,y,x) to (x',y',z'), where

$$x'=2x,\ y'=y/2 \quad \text{for } x \le \tfrac{1}{2},$$
$$x'=2-2x,\ y'=1-y/2 \quad \text{for } x > \tfrac{1}{2} \tag{2a}$$

is a Baker's transformation that performs the stretch-fold operation in the x,y plane, and

$$z'=z+\alpha(y'-\tfrac{1}{2})(\mathrm{mod}\,1) \tag{2b}$$

supplies the shear, where α is a nonnegative real number specifying the amount of shear. One application of this map constitutes the phase of rapid motion referred to previously. The effect of this map on the unit cube is shown in Fig. 1. For the purposes of our calculations, we assume that this map is repeated periodically in space, so that the field can be represented as a Fourier series.

We now regard the map as an instantaneous deformation of an electrically conducting material. The SFS map is homogeneous in z, and has the property that the set of material lines parallel to the x axis is mapped into itself. We can therefore restrict attention to magnetic fields of the form $\mathbf{B}(x,y,z)=e^{2\pi i z}b(y)\hat{\mathbf{x}}$, for complex-valued scalar functions $b(y)$, as this form is preserved under the map for arbitrary shear α.

Since the map is instantaneous, the material behaves as one of infinite conductivity, and the magnetic field immediately after the map is just the geometric deformation of the field immediately before. That is, $\mathbf{B}'(\mathbf{r}')=J(\mathbf{r})\mathbf{B}(\mathbf{r})$, where $\mathbf{r}'$ is the image of $\mathbf{r}$, and J is the Jacobian (derivative) matrix of the map. In terms of the function $b(y)$, we have

$$b'(y')=2\,\mathrm{sgn}(\tfrac{1}{2}-y')b(y), \tag{3}$$

where $y=\min(2y',2-2y')$. After the instantaneous mapping, the material is held still for a time T while the field diffuses within it. If

$$b'(y)=\sum_{j=-\infty}^{\infty} b_j' e^{2\pi i j y},$$

the effect of diffusion is to damp the jth coefficient by a factor $D(j)=\exp[-\epsilon(j^2+1)]$. Here, $\epsilon=4\pi^2/R$, and $R\equiv L^2/\eta T$ is the magnetic Reynolds number of the system. Thus, if we include both the mapping and diffusive effects, the field after $m+1$ steps is given in terms of the field after m steps by

$$b_j(m+1)=D(j)\sum_{k=-\infty}^{\infty} G(j,k)b_k(m), \tag{4}$$

where

$$G(j,k)=\frac{i}{\pi}\left[\frac{(-1)^j-e^{i\pi\alpha}}{j+\alpha-2k}+\frac{(-1)^j-e^{-i\pi\alpha}}{j+\alpha+2k}\right] \tag{5}$$

is the matrix that represents (3) on the Fourier coefficients of $b(y)$.

The representation (4) is well suited to numerical study with a truncated vector of Fourier coefficients $(b_{-N},\ldots,b_N)$. We simulated the field evolution by performing a number M of iterations of (4), using the initial condition $b_0=1$, $b_{j\neq 0}=0$, and choosing M sufficiently large that the coefficient vector essentially converged to the dominant eigenvector (or space) of (4). The quantity we used to measure dynamo action was the instantaneous mean-field growth rate, here defined to be $p_{\text{inst}}(M)=\ln|b_0(M)/b_0(M-1)|$. In practice, we found that $p_{\text{inst}}(M)$ had converged to a number p_{inst} with an accuracy of at least two and usually more significant figures after $M=40$ iterations, for values of α greater than 0.3 or so. By successively doubling N, we found that $N=64$, 128, and 256 were sufficiently large to obtain the same accuracy when ϵ took the values 0.01, 0.001, and 0.0001, respectively. In all cases, a rapid diffusive decay of the spectrum occurs prior to the cutoff at N.

For shear parameter α below 0.3 or so, the accuracy of the calculations was sharply diminished. The reason for the unreliability of the calculations at small α reflects the complexity of the field dynamics. At small ϵ, the spatial field develops oscillations of amplitude 2^m on a length scale 2^{-m} after m iterations. The dominant eigenmode, which is expected to be weakly decaying in time, therefore becomes very difficult to extract from the background noise. As we are primarily interested in the growing modes at $\alpha \ge 0.4$, we have not attempted to im-

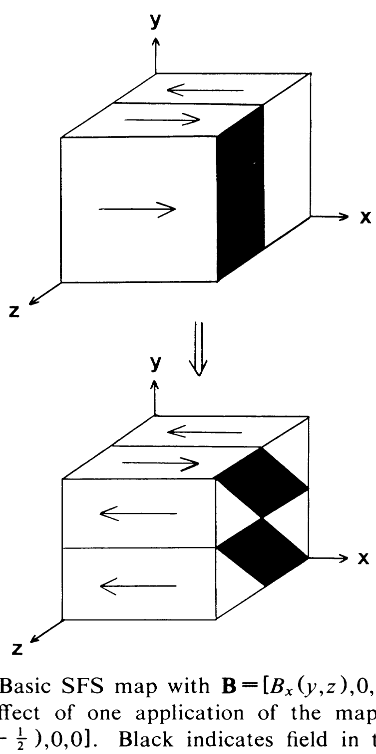

FIG. 1. Basic SFS map with $\mathbf{B}=[B_x(y,z),0,0]$. We illustrate the effect of one application of the map on the field $\mathbf{B}=[\mathrm{sgn}(z-\frac{1}{2}),0,0]$. Black indicates field in the positive x direction, white in the negative x direction.

prove the low-α calculations.

In Fig. 2, we plot the dominant growth rate $p_{\rm inst}$ as a function of α for $\epsilon=0.01$, 0.001, and 0.0001, respectively. For $\alpha>0.4$, and especially in the range $0.8<\alpha<1.2$, $p_{\rm inst}$ appears to approach a positive limit as $\epsilon\rightarrow 0$, strongly indicating fast-dynamo action. We can check this statement in a number of ways; we illustrate using the particular values 0.5 and 0.95 for α. When $\epsilon=0.0001$, we have $p_{\rm inst}(0.5)=0.04$ and $p_{\rm inst}(0.95)=0.2710$ after forty steps. If we set $\epsilon=0$ in (4), we obtain $p_{\rm inst}(0.5)=0.02$ and $p_{\rm inst}(0.95)=0.2702$ using an $N=256$ truncation. Another check is to calculate the field evolution *exactly* with the spatial equation (3). After m iterations, the mean field is the sum of 2^m unimodular complex numbers with highly irregular phases, and it is more meaningful to consider the average growth rate $p_{\rm ave}(m)=m^{-1}\ln|b_0(m)/b_0(0)|$. This sum is a tedious computation even for moderate values of m, but for $m=17$ we obtain average growth rates of 0.05854 and 0.2715 for $\alpha=0.5$ and 0.95, respectively. Although not rigorous, these checks support the claim that at $\alpha=0.95$ we are dealing with a fast dynamo. Fast-dynamo action is also suggested for $\alpha=0.5$, but the small-scale fluctuations alluded to above prevent reliable estimates at exactly zero diffusivity.

A clue to the reason why the dynamo growth rates are so insensitive to the nature of the high-wave-number cutoff can be found in the power spectra (not shown) of the eigenmodes found as final states of the iteration process. The spectra have very different diffusive tails, but their low-wave-number structures are almost identical. This indicates that the large-scale structures determine the mean-field growth, while diffusion sets the dissipation length scale for the largely passive small-scale magnetic structures.

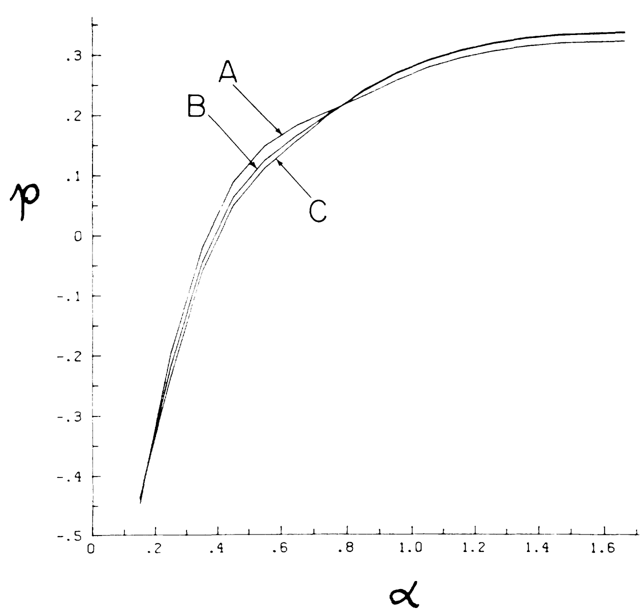

FIG. 2. Growth-rate p vs shear α for various ϵ, computed from the N-term Fourier truncation of Eq. (4). (A) $\epsilon=0.01$, $N=64$, (B) $\epsilon=0.001, N=128$, and (C) $\epsilon=0.0001, N=256$.

Although the SFS map (2) is discontinuous on the plane $y=\frac{1}{2}$, simple analytic flows act on the magnetic field in a similar way. Consider the effect of a flow with velocity field

$$\mathbf{u}(x,y,z)=\delta^{-1}(0,\tfrac{1}{2}\beta\sin 2\pi x,\tfrac{1}{2}\alpha\cos 2\pi x), \qquad (6)$$

acting for a time δ, followed by a change of coordinates $(x,y,z)\rightarrow(y,x,z)$. In the limit $\delta\rightarrow 0$ we obtain the map

$$(x',y',z')=(y+\tfrac{1}{2}\beta\sin 2\pi x,x,z+\tfrac{1}{2}\alpha\cos 2\pi x). \qquad (7)$$

For β near 1, the action of this map in the x,y plane resembles the Baker transformation part of the SFS map; that is, it essentially doubles the lengths of, and roughly preserves the orientation of, lines aligned with the x direction. The z component of (7) mimics the shear part; note that α plays essentially the same role as in the SFS map. The senses of the stretch-fold-shear motions in this flow change sign between adjacent subcubes of side $\frac{1}{2}$, so that the pulsed flow resembles the action of an antisymmetric periodic extension of the SFS map.

To maximize the similarity between the pulsed-flow model and the most successful SFS dynamos, we choose $\alpha=\beta=1$ and use only these values in subsequent calculations. This choice makes (6) a *Beltrami* flow, i.e., a flow in which the velocity and vorticity are everywhere parallel. We remark that Beltrami flows have long played a role in dynamo theories[12]; indeed, the steady flow in Soward's calculations was a superposition of two Beltrami flows of the form (6).

The evolution of the magnetic field for one mapping-diffusion time step is simply calculated, although we now need to keep track of the two vector components

$$(B_x,B_y)=e^{2\pi iz}\sum_{j,k=-\infty}^{\infty}(Bx_{jk},By_{jk})\exp 2\pi i(jx+ky).$$

The mapping of the Fourier coefficients is given by a matrix equation resembling (4), except that the matrix elements now involve Bessel functions rather than the simple rational and trigonometric functions appearing in (5). The field evolution is again simulated by restriction of the indices j,k to the range $-N,\ldots,N$ and iteration of the matrix equation. Our preliminary numerical results suggest that the analogy with the SFS map is borne out and the fast-dynamo activity is obtained for α and β equal to 1. For $N=16$ and 40–50 iterations, we obtain instantaneous growth rates in the range 0.28–0.33 at $\epsilon=0.04$ and 0.44–0.47 at $\epsilon=0.004$. With $N=32$, we also find a growth rate in the range 0.44–0.48 at $\epsilon=0.0004$.

Since the stretching of line elements by a factor of approximately 2 is a common feature of both the SFS map and the pulsed flow with $\beta=1$, these growth rates indicate that the pulsed flow is the more efficient dynamo. The reasons for this apparent efficiency are unclear, but it seems likely that they are associated with the considerable inhomogeneity of the stretching and shearing actions in the flow when compared with the SFS map.

The dynamo growth rates in both the SFS and pulsed-flow models are insensitive to ϵ as $\epsilon \rightarrow 0$, despite the fact that the corresponding eigenfunctions develop more and more complicated spatial structure as $\epsilon \rightarrow 0$. Apparently, growth of the mean field is largely unaffected by the cascade of magnetic energy to the small scales where it is eventually dissipated. This is not surprising on physical grounds, and can actually be demonstrated for the SFS dynamo.

At exactly zero diffusivity, the energy of the magnetic field quadruples with every iteration of the SFS map. Mathematically, this implies that the matrix $G/2$ [defined by Eq. (5)] is an isometry on the vector space of Fourier coefficients $(\ldots, b_{-1}, b_0, b_1, \ldots)$ equipped with the energy norm. Now, the eigenvalues of isometries necessarily have unit magnitude, as long as the corresponding eigenvectors have finite norm. Since the dynamo growth rates do not approach $\ln(2)$ as $\epsilon \rightarrow 0$, we surmise that the corresponding limiting eigenvectors have infinite energy norm. This behavior is exhibited in simpler models also; consider, for example, the operator H defined by $Hf(k) = \exp(-\epsilon k^2) f(k/2)$. For ϵ exactly zero, $H/\sqrt{2}$ is an isometry on the space of square-integrable functions, while it is easily verified that for all positive ϵ, the only eigenvalues have modulus 1.

A more suitable norm in which to study the convergence of eigenfunctions of the SFS dynamo is defined by

$$\|b\|^2 \equiv \sum_{-\infty}^{\infty} f(j) |b_j|^2, \tag{8}$$

where $f(j \neq 0) = |j|^{-\gamma}$, $f(0)=1$, and γ is a real number between 1 and 2. It can be shown that the error incurred by the N-term Fourier truncation of the full problem (4) is less than $C \ln N/N$ in the norm (8), where C is a number independent of N. This estimate holds uniformly as $\epsilon \rightarrow 0$, so that as far as this norm is concerned the limit $\epsilon \rightarrow 0$ can be studied by setting $\epsilon = 0$ explicitly and choosing N sufficiently large. Since the mean-field growth rate is independent of the choice of norm, we can expect the numerically observed growth rates to converge as $N \rightarrow \infty$, regardless of the value of ϵ.

The present paper has examined dynamo action in a class of unsteady three-dimensional flow models. We have confirmed that chaotic flow with positive Liapunov exponent is now sufficient by itself for a fast dynamo, but in conjunction with a shear that rearranges regions of opposite flux, chaotic line stretching can result in highly robust fast-dynamo action. The magnetic fields arising from flows with SFS behavior have exceedingly complex structure, but both the numerical and analytical results show that the dynamo is essentially determined by the large scales, with the small scales just cascading passively to ever smaller scales.

Our model relies heavily on the splitting of the flow into a brief interval of fast motion and a rest period of diffusion. While highly idealized, this picture allows the conceptual separation of the advective and diffusive effects, and indeed is quite a reasonable model for flows in which the characteristic diffusion time is large compared to the time scale for deformation of the conducting material. The use of this type of model by Parker[10] and Backus[11] led to important advances in understanding magnetic processes in conducting flows. We hope that our models also will lead to better understanding of the dynamo problem in highly conducting fluids.

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