Equivalence between γ Instability and Rigid Triaxiality in Finite Boson Systems

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It is shown, by use of the O(6) limit of the interacting-boson model, that the γ-unstable state can be generated from the rigid triaxial intrinsic state with $\gamma=30°$. This equivalence between the two descriptions holds under a certain condition on the number of bosons, which is satisfied in realistic cases. The β-γ deformation potential calculated from this triaxial intrinsic state has a minimum also at $\gamma=30°$, though quite shallow. The validity of the potential and the triaxial intrinsic state, which has been questioned because of the shallow minimum, is thus confirmed.

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The triaxial deformation has been one of the most interesting and important subjects in the study of nuclear structure, because it determines the shape of the nuclear surface. Since a γ-unstable model of Wilets and Jean,[1] and a rigid-triaxial-rotor model of Davydov and Filippov,[2] the triaxial deformation has provided various questions, some of which are still open. One of the major open questions is the relationship between physical pictures contained in the above two models. The γ-unstable picture assumes that the wave function is distributed over a wide range of γ variable, which means that the nucleus is soft with respect to the triaxial shape. On the other hand, the wave function in the rigid-triaxial-rotor picture has a sharp peak at a finite point of γ variable, indicating a stable or static triaxial deformation. The two models are thus formulated on different physical pictures, although these models have been applied sometimes to the same nuclei.

We shall consider in this Letter the relationship between these two pictures from the viewpoint of the interacting-boson model (IBM).[3] The O(6) limit[4] of the IBM is known[5,6] as a typical example of γ-unstable systems. As shown by Ginocchio and Kirson,[5] the O(6) Hamiltonian can be equated with the γ-unstable Bohr-Mottelson[7] Hamiltonian. This consequence has been obtained in the intrinsic-state formalism. The IBM intrinsic wave function including the γ degree of freedom is defined generally as[5,6]

$$|N,\beta,\gamma\rangle=(N!)^{-1/2}[\lambda^\dagger(\beta,\gamma)]^N|0\rangle, \tag{1}$$

where N stands for the number of bosons, and

$$\lambda^\dagger(\beta,\gamma)=(1+\beta^2)^{-1/2}[s^\dagger+\beta\cos\gamma d_0^\dagger+\sqrt{\tfrac{1}{2}}\beta\sin\gamma(d_2^\dagger+d_{-2}^\dagger)], \tag{2}$$

with β and γ being parameters. Ginocchio and Kirson have shown that the O(6) states of $\sigma=N$ can be written as[5]

$$|[N],\sigma=N,\tau,v_\Delta,L,M\rangle=\int_0^{\pi/3}d\gamma\sin3\gamma\int d\Omega\,\bar{\Psi}^{\tau,v_\Delta}_{L,M}(\gamma,\Omega)\hat{R}(\Omega)|N,\beta=1,\gamma\rangle, \tag{3}$$

where L and M are the angular momentum and its z component, σ, τ, v_Δ are quantum numbers in the O(6) limit, $\bar{\Psi}$ denotes an amplitude, Ω means the Euler angle, and $\hat{R}(\Omega)$ is the usual rotation operator. In the case of the ground state, this turns out to be

$$|[N],\sigma=N,\tau=0,v_\Delta=0,L=0\rangle=\int_0^{\pi/3}d\gamma\sin3\gamma\int d\Omega\,\hat{R}(\Omega)|N,\beta=1,\gamma\rangle. \tag{4}$$

The second integral simply works as the projection onto 0^+. The first integral ranges over $\gamma=0-\pi/3$, indicating the γ-unstable nature of the O(6) limit. The O(6) limit is thus considered to be an ideal example of the γ-unstable system. Meyer-Ter-Vehn has reported a similar conclusion in terms of energies and $B(E2)$ values.[8]

On the other hand, the rigid-triaxial-rotor aspect of the O(6) limit has been pointed out by Casten, Aprahamian, and Warner,[9] Castanos, Frank, and Van Isacker,[10] Dobes,[11] and Elliott, Evans, and Van Isacker[12] from various viewpoints. Dobes calculated a β-γ potential,[11]

$$V(\beta,\gamma)=\frac{\langle N,\beta,\gamma|HP_{I=0}|N,\beta,\gamma\rangle}{\langle N,\beta,\gamma|P_{I=0}|N,\beta,\gamma\rangle}, \tag{5}$$

where P_I denotes the projection operator onto the angular momentum I, and H is an IBM Hamiltonian. For an O(6) Hamiltonian, a minimum of $V(\beta,\gamma)$ is found not only in the β direction, but also in the γ direction at $\gamma=30°$.[11] This minimum should be an indication of the rigid triaxial deformation, if the minimum is deep enough. There are thus two conflicting pictures, γ unstable and rigid triaxial rotor, for the O(6) limit. We shall resolve this problem.

The rigid triaxiality can be introduced to the IBM by calculating physical quantities through the intrinsic state (1) with a fixed γ. We here take a Hamiltonian

$$H=-\kappa Q\cdot Q,\quad \text{with } Q=d^\dagger s+s^\dagger\tilde{d}, \tag{6}$$

where κ is a parameter, and $\tilde{d}_m=(-)^m d_{-m}$. This Hamiltonian yields an O(6) solution.[9,11] Figure 1(a) shows the potentials $V(\beta=\frac{1}{2},\gamma)$, $V(\beta=1,\gamma)$, and $V(\beta=\sqrt{2},\gamma)$ in (5) as functions of γ for the Hamiltonian (6) with $\kappa=0.05$(MeV) and $N=5$. The potential minimum is indeed found at $\beta=1$ and $\gamma=30°$. The projected wave function in (5) is denoted as $\Phi(N,\beta,\gamma)=\mathcal{N}P_{I=0}|N,\beta,\gamma\rangle$, where $\mathcal{N}$ is a normalization constant.

In order to see the structure of $\Phi(N,\beta,\gamma)$, the overlap between this state and the O(6) state is evaluated,

$$X(N,v_\Delta,\beta,\gamma)=([N],\sigma=N,\tau=3v_\Delta,v_\Delta,L=0\,|\,\Phi(N,\beta,\gamma)). \tag{7}$$

Figure 1(b) shows the square of this overlap for $v_\Delta=0$ and $\beta=\frac{1}{2}$, 1, and $\sqrt{2}$ as a function of γ. Note that the O(6) ground state has $v_\Delta=0$. The overlap in Fig. 1(b) becomes perfect (i.e., unity) at $\beta=1$ and $\gamma=30°$. This implies that, although the O(6) wave function is γ unstable, it can be generated from a rigid triaxial intrinsic wave function by the angular-momentum projection, if β and γ are appropriate. In other words, the γ unstable and rigid triaxial descriptions can be equivalent in the sense that they have the same wave function. To see this point more precisely, we expand the projected state as

$$\Phi(N,\beta,\gamma)=\mathcal{N}'\textstyle\sum_{v_\Delta}C_{N,v_\Delta}P_{v_\Delta}(\cos3\gamma)\,|\,[N],\sigma=N,\tau=3v_\Delta,v_\Delta,L=0), \tag{8}$$

where $\mathcal{N}'$ is a normalization constant, P_L stands for the Lth-order Legendre polynomial, and

$$C_{N,v_\Delta}=\left[\frac{2v_\Delta+1}{(N-3v_\Delta)!(N+3v_\Delta+3)!}\right]^{1/2}$$

For $3\le N\le 5$, v_Δ can be 0 or 1. If γ is chosen so that $P_1(\cos3\gamma)=0$, nonvanishing amplitude on the right-hand side of Eq. (8) is only for the $v_\Delta=0$ component which is nothing but the exact O(6) ground state. Thus, for $N\le 5$, $\Phi(N,\beta=1,\gamma=30°)$ is identical to the exact O(6) ground state. For larger N, this identity is lost in principle, but remains in a rather good approximation. Figure 2 shows the probability $[X(N,v_\Delta,\beta=1,\gamma=30°)]^2$ for $v_\Delta=0$, 2, and 4 as a function of N. Because of $\gamma=30°$, $v_\Delta=1,3,5,\ldots$ are not contained in $\Phi(N,\beta=1,\gamma=30°)$. For realistic N ($\lesssim 10$), the wave function $\Phi(N,\beta=1,\gamma=30°)$ is completely or nearly comprised of the $v_\Delta=0$ component. For example, at $N=10$, the $v_\Delta=0$ component accounts for more than 99% of the projected wave function, whereas the $v_\Delta=2$ component occupies less than 1%. Only at $N>10$ does the $v_\Delta=2$ component contribute practically. At $N\to\infty$, however, various v_Δ become equally important in $\Phi(N,\beta=1,\gamma=30°)$, and the integration with respect to γ in (4) is essential to obtain the O(6) ground state. On the other hand, the integration is redundant for smaller N, while it still yields the correct result.

The rigid triaxial intrinsic wave function can thus produce the γ-unstable wave function in systems with a realistic number of bosons. This has been quite unexpected because of the shallow minimum of the potential. For instance, Dobes pointed out[11] that the potential in (5) is too shallow to correspond to the static triaxial rotor.

We have so far considered the ground state. Excited states can be treated similarly. To obtain the 2^+ state, for instance, $K^\pi=0^+$ and 2^+ components are extracted in the usual way from the intrinsic state $|N,\beta=1,\gamma=30°)$, and are projected onto $J^\pi=2^+$. The projected states are not orthogonal because of K mixing. The Hamiltonian is then diagonalized in the subspace

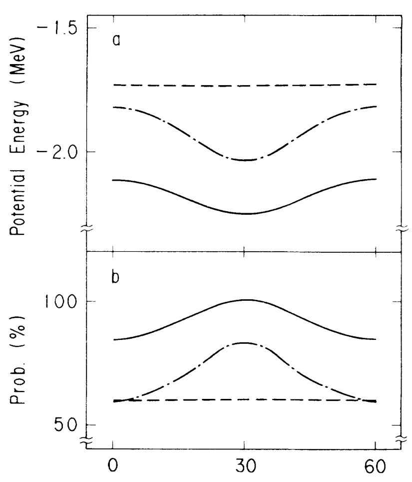

FIG. 1. (a) Deformation potential $V(\beta,\gamma)$ for $N=5$. The solid line is for $\beta=1$, while the dashed and dashed-dotted lines are for $\beta=\frac{1}{2}$ and $\sqrt{2}$, respectively. (b) Square of the overlap for $N=5$ between the O(6) wave function and the $\beta=1$ and $\gamma=30°$ projected wave function. Three lines correspond to three values of β as in (a).

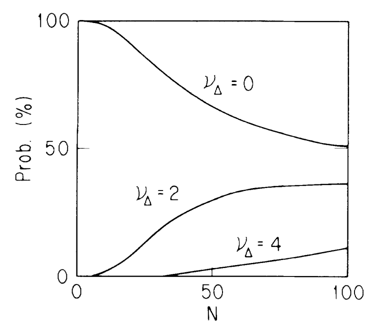

FIG. 2. Square of the overlap between the O(6) wave function and the $\beta=1$ and $\gamma=30°$ projected wave function as function of the boson number, N.

spanned by these nonorthogonal states, and (approximate) eigenenergies are obtained. For the ground state, only $K^\pi=0^+$ is possible, and hence there is no need for the orthogonalization. The resultant state is nothing but $\Phi(N,\beta=1,\gamma=30°)$. For $J^\pi>2^+$, one has to diagonalize the Hamiltonian in a larger subspace with $K^\pi=0^+,2^+,\ldots,J^+$, while the basic formalism remains unchanged.

As an example of the above calculation, we shall consider ^{196}Pt, which exhibits the typical O(6) patterns, and has been analyzed in terms of the O(6) limit.[4] The parameters of the O(6) Hamiltonian have already been adjusted[4] so as to fit the experimental data. The calculated excitation energies of $\sigma=N$ states are shown in Fig. 3. Note that these calculated excitation energies are in good agreement with experimental ones.[4] By the projection method outlined above, we calculate energies from the same Hamiltonian, assuming $\beta=1$ and $\gamma=30°$. This spectrum is also shown in Fig. 3. The agreement between the two theoretical spectra is nearly perfect. The energies of nine states out of fifteen states shown in Fig. 3 are reproduced exactly by the projection calculation. The largest discrepancy is 42 keV at the 4_2^+ state, of which the excitation energy is 1100 keV. The remarkable agreement in Fig. 3 suggests that the rigid $\gamma=30°$ triaxial description holds not only for the ground state, but also for the excited states. For N larger, this agreement becomes less accurate, similarly to the $v_\Delta=0$ overlap in Fig. 2.

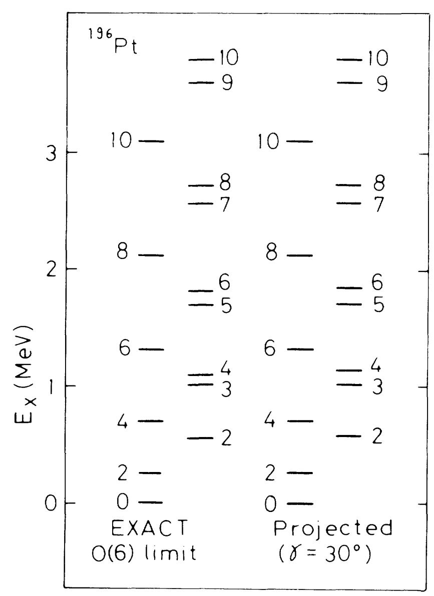

FIG. 3. The exact O(6) spectrum compared to the spectrum calculated by the projection of the rigid triaxial intrinsic state. Numbers beside the energy levels indicate the angular momenta. The nucleus is ^{196}Pt, and the Hamiltonian is taken from Ref. 4.

One can carry out the variation with respect to β and γ for each state separately in a more sophisticated calculation, although it should not make much difference in the present case. Such improved variation may be needed to treat higher bands which are not shown in Fig. 3.

The Wilets-Jean model[1] and the Davydov-Filippov model[2] are formulated basically for the classical limit, which means $N\to\infty$ in the case of IBM.[5,6] The rigid triaxial nuclear shape is assumed in the Davydov-Filippov model,[2] whereas the nuclear shape is extremely soft in the Wilets-Jean model.[1] The two models are thus based on different pictures of the nuclear shape. These two pictures, however, give rise to the same wave function in systems with finite number of particles. This equivalence suggests that, in such finite systems, one can utilize the γ variable as a useful tool to describe collective states, but cannot introduce the concept of the intrinsic triaxial nuclear surface, irrespectively of soft or rigid. This should be an effect of finiteness of the system. The situation is, in fact, reversed in the classical limit, $N\to\infty$. Although the situation at $N\to\infty$ will be discussed in detail in a forthcoming paper, it should be mentioned that the $N\to\infty$ limit of the present intrinsic state shows a pattern of the energy levels identical to that of the Davydov-Filippov model. In other words, the intrinsic state is connected at $N\to\infty$ to the Davydov-Filippov model.

We have considered so far boson systems only. The above equivalence, however, is not limited to bosons. In fact, one can derive the same conclusion for the SO(6) case of the Ginocchio model[13] which is purely a fermionic model.

The deformation potential in the β-γ plane has been widely used in BCS+Nilsson-type calculations with axial asymmetry, where a potential minimum is often found at $\gamma\neq0°$.[14] Although this minimum is indeed shallow particularly in the γ direction, the validity of the rigid triaxial intrinsic state has been simply assumed rather than examined. The present result seems to give the first strong support to this assumption. In fact, the axially symmetric Nilsson wave function has close similarity to the axially symmetric IBM intrinsic state[15] and it is likely that one can extend this similarity to axially asymmetric cases. Further studies are in progress along this direction.

It is obvious that the number of particles is a crucial quantity in the above discussions. The number of bosons is fixed in the IBM by the number of valence nucleon pairs.[16] If the boson number were set very large, we should have been lead to a completely different conclusion. The importance of the finite boson number should be emphasized.

Until now, we have considered the O(6) limit. The rigid triaxial intrinsic state provides a good approximation to the exact solution also in situations away from the O(6) limit, as far realistic cases are concerned. The appropriate value of γ is then determined by the variation for the potential (5). The resultant γ becomes smaller in going further away from the O(6), for instance $\gamma=0°$ in the SU(3) limit.[17] This consequence again supports the BCS+Nilsson-type calculation with axial asymmetry for transitional nuclei. Note that the angle γ has to be determined after the angular-momentum projection. Otherwise, one may obtain completely ridiculous results. All details will be presented in a forthcoming paper.

In conclusion, we emphasize once more that, in finite systems, the triaxial shape may not be a well-defined concept, and the soft and rigid triaxial descriptions can be identical. Because of this identity, a single triaxial intrinsic wave function is useful in describing low-lying collective states including ones with large deviations from the axial symmetry, although the deformation potential may have only a shallow minimum. It is very important practically that one can replace the γ-unstable description with the rigid triaxial description, because the latter is much easier to handle. In fact, the γ-unstable system is not easy in general to describe, because the superposition over a wide range of the γ variable is required as shown in (3). The IBM, including transitional situations, is a precious system where the γ instability can be analyzed exactly in a quantum-mechanical way for finite systems. The equivalence *Ansatz* introduced in this Letter should be studied further in various nuclear many-body systems.

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