Domain Walls of Finite Thickness in General Relativity

A. K. Raychaudhuri and Gangabrata Mukherjee
Physics Department, Presidency College, Calcutta 700073, India
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The case of a domain wall of finite thickness is examined with use of two different expressions for the energy-stress tensor of domain walls but no symmetry condition. It is found that the equations of general relativity show that such walls cannot be in static equilibrium, and further even nonequilibrium walls can be consistent with Einstein's equations only under very special circumstances.

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The study of domain walls in general relativity has led to some rather intriguing results. Following Zel'dovich, Kobzarev, and Okun',[1] if we consider a Lagrangean of a scalar field ϕ in the form

$$\mathcal{L}=\tfrac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}-\lambda^2(\phi^2-\eta^2)^2, \tag{1}$$

then the classical field equations are

$$g^{\alpha\beta}\phi_{,\alpha;\beta}-(\partial/\partial\phi)[\lambda^2(\phi^2-\eta^2)^2]=0. \tag{2}$$

In the broken-symmetry state, one may have two regions where $\phi=\pm\eta$ separated by a layer whose thickness at rest is small but nonvanishing. Obviously the energy-stress tensor vanishes in the regions where ϕ has the constant value $\pm\eta$. However, in the intervening layer, called the domain wall, it is not so. To evaluate the energy-stress tensor in the region, suppose that $\phi_{,\alpha}$ in the transition layer is a spacelike vector. Then we can choose locally Lorentz coordinates, such that $\partial\phi/\partial x^0=\partial\phi/\partial x^2=\partial\phi/\partial x^3=0$, where x^0 is the time coordinate and x^2,x^3 are locally tangential to the wall surface. The energy-stress tensor has then the components $T^0_0=T^2_2=T^3_3=\sigma$ (say), $T^1_1=0$, and all nondiagonal components vanish. Here the result $T^1_1=0$ is obtained from the divergence relation and the boundary condition (cf. Vilenkin[2]). We can give a tensorial form to the energy-stress tensor by making a general transformation (cf. Ipser and Sikivie[3]):

$$T^\alpha{}_\beta=\sigma(\delta^\alpha{}_\beta+\xi^\alpha\xi_\beta), \tag{3}$$

where ξ^α is a unit spacelike vector orthogonal to the wall surface. The expression (3) has been obtained on the assumption that $\phi_{,\alpha}$ is spacelike or, in other words, that one can introduce a time coordinate in which the ϕ field is stationary. This would not be possible if $\phi_{,\alpha}$ were timelike.

This form for the energy-stress tensor may not be correct for thick domain walls as T^1_1 even though vanishing at the boundaries may not vanish throughout the wall. The form (3) should therefore be taken as an *ad hoc* generalization of the thin-wall case.

While the expression (3) has been used in all previous studies (Vilenkin,[2] Ipser and Sikivie,[3] Tomita,[4] and Linet[5]), those studies have considered the domain wall to be a surface of vanishing thickness. But the thickness of the domain wall has a finite value ($\Delta\simeq 1/m_x\simeq 1/\lambda\eta$) though small. In the present note we shall consider the possibility of a finite thickness of the wall and proceed to show that the energy-stress tensor (3) is inconsistent with the equations of general relativity in the static case.

For the Lagrangean (1), the energy-stress tensor can be written as

$$T^\alpha{}_\beta=g^{\alpha\gamma}\phi_{,\gamma}\phi_{,\beta}-\delta^\alpha{}_\beta[\tfrac{1}{2}g^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta}-\lambda^2(\phi^2-\eta^2)^2]. \tag{4}$$

Contracting both (3) and (4) with ξ_α, we get

$$(\xi^\gamma\phi_{,\gamma})\phi_{,\beta}-\xi_\beta[\tfrac{1}{2}g^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta}-\lambda^2(\phi^2-\eta^2)^2]=0. \tag{5}$$

Equation (5) shows that ξ_α is hypersurface orthogonal. We proceed to show that ξ_α is also geodetic and divergence free. Taking the divergence of (3), we get

$$\sigma_{,\beta}+\sigma_{,\alpha}\xi^\alpha\xi_\beta+\sigma\xi^\alpha_{;\alpha}\xi_\beta+\sigma\xi_{\beta;\alpha}\xi^\alpha=0. \tag{6}$$

On contraction with ξ^β, Eq. (6) reduces (provided $\sigma\neq 0$) to

$$\xi^\alpha_{;\alpha}=0. \tag{7}$$

Also by comparison of (3) and (4),

$$\sigma=2\lambda^2(\phi^2-\eta^2)^2. \tag{8}$$

Hence, we may write

$$\sigma_{,\alpha}=\mathcal{N}\xi_\alpha, \tag{9}$$

where $\mathcal{N}$ is a scalar. Any arbitrary vector A^α can be split up as $A^\alpha=B^\alpha+\psi\xi^\alpha$, where B^α is normal to ξ^α and, hence, to $\sigma_{,\alpha}$ as well. Contracting (6) with A^α, we get

$$\sigma\xi_{\beta;\alpha}\xi^\alpha A^\beta=0, \tag{10}$$

i.e., ξ_α is geodetic. Also the hypersurface orthogonality and geodesicity of ξ^α and the constancy of $\xi^\alpha\xi_\alpha$ give

$$\xi_{\alpha;\beta}-\xi_{\beta;\alpha}=0. \tag{11}$$

Now consider the identity

$$(\xi^\alpha_{;\beta\alpha}-\xi^\alpha_{;\alpha\beta})\xi^\beta\equiv -R_{\alpha\beta}\xi^\alpha\xi^\beta.$$

As the vector ξ^α is hypersurface orthogonal, divergence free, and geodetic, one can obtain the following scalar equation (the procedure is same as that for the derivation of the Raychaudhuri equation):

$$S_{\alpha\beta}S^{\alpha\beta} - R_{\alpha\beta}\xi^{\alpha}\xi^{\beta} = 0, \tag{12}$$

where $S_{\alpha\beta}$ is defined as

$$S_{\alpha\beta} = \xi_{(\alpha;\beta)} - \tfrac{1}{3}(g_{\alpha\beta} + \xi_{\alpha}\xi_{\beta})\xi^{\gamma}_{;\gamma} + \dot{\xi}_{(\alpha}\xi_{\beta)}. \tag{13a}$$

Note that the second and the third terms on the right-hand side in Eq. (13a) vanish and, in view of Eq. (12), one can write simply

$$S_{\alpha\beta} = \xi_{\alpha;\beta}. \tag{13b}$$

With use of Einstein's field equations and the expression for the energy-stress tensor $T^{\alpha}{}_{\beta}$ from Eq. (3), Eq. (12) reduces to

$$S_{\alpha\beta}S^{\alpha\beta} + 12\pi\sigma = 0. \tag{14}$$

Obviously Eq. (14) can be satisfied only if $S_{\alpha\beta}S^{\alpha\beta} < 0$. However, if the eigenvalues ρ_a of $S_{\alpha\beta}$ are all real, $S_{\alpha\beta}S^{\alpha\beta} = \sum_a \rho_a^2 > 0$.[6] Hence, the validity of (14) demands that there must be a pair of complex conjugate eigenvalues. Writing these as $C \pm iD$ (the other two eigenvalues would be $-2C$ and 0, because $S^{\alpha}_{\alpha} = 0$ and $S_{\alpha\beta}\xi^{\alpha} = 0$), we have

$$S_{\alpha\beta} = (C+iD)l_{\alpha}l_{\beta} + (C-iD)m_{\alpha}m_{\beta} + 2C\eta_{\alpha}\eta_{\beta}, \tag{15}$$

where l^{α}, m^{α}, η^{α}, and ξ^{α} constitute an orthonormal tetrad with l^{α} and m^{α} complex:

$$l^{\alpha} = (t^{\alpha} + i\kappa^{\alpha})/\sqrt{2}, \quad m^{\alpha} = (t^{\alpha} - i\kappa^{\alpha})/\sqrt{2}, \tag{16}$$

t^{α} and x^{α} being unit timelike and spacelike real vectors, orthogonal to one another and to η^{α} and ξ^{α}. With Eqs. (15) and (16), we have

$$S_{\alpha\beta}S^{\alpha\beta} = 6C^2 - 2D^2,$$

and Eq. (14) becomes

$$D^2 = 3C^2 + 6\pi\sigma. \tag{17}$$

We have not been able to find out the situation under which the rather stringent conditions (15) and (17) may hold. However, it is easy to see that in case there is a timelike hypersurface-orthogonal Killing vector (i.e., the wall is in static equilibrium), $S_{\alpha\beta}S^{\alpha\beta} > 0$ and Eq. (14) cannot be satisfied. If K^{α} be that Killing vector, then the Lie derivation of $T^{\alpha}{}_{\beta}$ (and, hence, T) with respect to K^{α} must vanish. In view of (3), this gives

$$\sigma_{,\alpha}K^{\alpha} = 0, \tag{18}$$

$$\xi_{\alpha;\beta}K^{\beta} = K_{\alpha;\beta}\xi^{\beta}. \tag{19}$$

In view of Eq. (9), (18) gives

$$\xi_{\alpha}K^{\alpha} = 0. \tag{20}$$

Also as K^{α} is hypersurface orthogonal,

$$K_{[\alpha}K_{\beta;\gamma]} = 0 \Longrightarrow \xi^{\alpha}K_{[\alpha}K_{\beta;\gamma]} = 0.$$

Written out explicitly and with use of Eq. (20), the above becomes

$$\xi^{\alpha}[K_{\beta}K_{\gamma;\alpha} + K_{\gamma}K_{\alpha;\beta}] = 0.$$

Again, because K^{α} is a Killing vector, the above equation gives with use of Eq. (19)

$$K_{\beta}K^{\alpha}\xi_{\gamma;\alpha} - K_{\gamma}\xi_{\beta;\alpha}K^{\alpha} = 0,$$

showing that K^{α} is an eigenvector of $\xi_{\gamma;\alpha}$ and, hence, of $S_{\gamma\alpha}$ because of Eq. (13b). It now follows that eigenvalues are all real and, hence, $S_{\alpha\beta}S^{\alpha\beta} \geq 0$, the equality occurring when $S_{\alpha\beta}$ vanishes.

The foregoing discussion apparently shows the inconsistency of the idea of a static thick domain wall if both the forms (3) and (4) for the energy-stress tensor are used along with Einstein's equations of general relativity.

A more fundamental point is the reasonableness of the application of classical general relativity and expressions for energy-stress tensor within a domain wall where quantum effects would obviously dominate. However, this question can be answered only when a satisfactory quantum theory of gravity is developed.

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