Arnol'd Diffusion in $1\frac{1}{2}$ Dimensions

D. W. Longcope and R. N. Sudan
Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14853
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A recent study by Zaslavskii *et al.* demonstrates that under certain conditions a charged particle may be accelerated by an arbitrarily small electrostatic wave packet in a magnetic field to arbitrary energy by the process of Arnol'd diffusion in $1\frac{1}{2}$ dimensions. A relativistic calculation shows, however, that the particles can only be accelerated up to a critical energy by a wave packet above a critical amplitude.

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It is well known that in Hamiltonian systems with three or more degrees of freedom there exists so-called Arnol'd diffusion[1] along a certain separatrix net covered by a "stochastic layer" that extends over all of phase space. Recently Zaslavskii *et al.*[2] have shown that in a particular lower-dimensional Hamiltonian system it is possible to have a separatrix net that spreads over the entire phase plane. They estimate the thickness Δ of the stochastic web into which the separatrix net is deformed to be of order $\Delta \sim \exp(-\mathrm{const}/\epsilon)$, where ϵ determines the magnitude of the time-dependent perturbation.

The system actually studied by Zaslavskii *et al.*[2] in detail is the motion of charged particles in a uniform magnetic field perturbed by an electrostatic wave packet propagating perpendicular to this magnetic field. This is an important problem relevant to plasma heating by high-frequency waves through cyclotron resonances and has been treated before by a number of authors[3-6] under a variety of different configurations and approximations. However, Zaslavskii *et al.*[2] come to an important new conclusion: For arbitrarily small wave electric field the particles can diffuse arbitrarily far into the region of high energies along the stochastic web. In this Letter we propose to examine this conclusion by introducing a relativistic treatment for the particle dynamics. In so doing we find that Zaslavskii *et al.*'s conclusion is correct only below a critical threshold for particle energies $mc^2\gamma_c$ given by Eq. (21) and even then it requires the wave-packet electric field to be above the threshold defined by (25). Thus, we maintain that the stochastic web exists for $\gamma < \gamma_c$ where the system may be treated as degenerate, but above γ_c the energy (action) dependence of the cyclotron frequency, Ω, becomes important, i.e., $\partial\Omega/\partial\gamma \neq 0$, and the system displays the normal properties of a Hamiltonian system governed by the Kolmogorov-Arnol'd-Moser (KAM) theorem.

The position (x,y) of a magnetized charged particle in a wave packet is determined by

$$\ddot{x} + \dot{x}\dot{\gamma}/\gamma + \Omega^2 x/\gamma^2 = (q/m\gamma)E(x,t), \tag{1}$$

$$\dot{y} = -(\Omega/\gamma)x, \tag{2}$$

where $\Omega = qB_0/mc$ is the nonrelativistic gyrofrequency and $\gamma \equiv [1-(\dot{x}/c)^2-(\dot{y}/c)^2]^{-1/2}$. Following Zaslavskii *et al.*,[2] we let the electric field be

$$E(x,t) = E\sum_{n=-\infty}^{\infty}\sin(kx - n\Delta\omega t) = ET\sin kx\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{3}$$

with $T = 2\pi/\Delta\omega$. Introducing (3) into (1) and (2), we observe that the particle motion is one of free gyration in between the impulses delivered by the wave packets at $t = nT$. This allows Eq. (1) to be written in the form of a mapping which connects the velocities at $t=(n+1)T-0$ with those at $t = nT - 0$. The relativistic equations of motion are given by

$$u_{n+1} = (u_n + K\sin v_n)\cos(\alpha/\gamma) + v_n\sin(\alpha/\gamma), \tag{4}$$

$$v_{n+1} = -(u_n + K\sin v_n)\sin(\alpha/\gamma) + v_n\cos(\alpha/\gamma), \tag{5}$$

$$\gamma = \{1+\beta^2[(u_n+K\sin v_n)^2 + v_n^2]\}^{1/2}, \tag{6}$$

where $u = k\gamma\dot{x}/\Omega$, $v = k\gamma\dot{y}/\Omega$, $K = -2\pi qEk/m\Omega\Delta\omega$, $\alpha = \Omega T = 2\pi\Omega/\Delta\omega$, and $\beta = \Omega/kc$. The nonrelativistic form of Eqs. (4) and (5) previously derived by Zaslavskii *et al.*[2] are obtained in the limit $\beta \rightarrow 0$.

Let the mapping described by (4) and (5) be denoted by $\mathcal{M}$. Following standard techniques based on the expansion, $\mathcal{M} = \mathcal{M}_0 + K\mathcal{M}_1 + K^2\mathcal{M}_2 + \ldots$, in powers of K, it is easy to see that a particle initially at $\boldsymbol{\rho}_0 = (u_0, v_0)$ will be rotated by the mapping $\mathcal{M}_0$ through an angle $\alpha/\gamma_0(u_0,v_0)$. If the particle is at a particular radius $\rho_0 = (u_0^2+v_0^2)^{1/2}$ such that $\alpha/\gamma_0 = 2\pi p/q$ for some incommensurate pair of p and q, then q iterations of $\mathcal{M}_0$ will return the particle to $\boldsymbol{\rho}_0$. Thus for a given choice of p and q

$$\rho_{p,q} = \beta^{-1}[(\alpha q/2\pi p)^2 - 1]^{1/2}, \tag{7}$$

the fixed points of $\mathcal{M}_0^q$ form a set of concentric circles at radii $\rho_{p,q}$, with $p = 1, 2, \ldots, [\alpha q/2\pi]$, where $[x]$ denotes the integer part of x. Of these points it is possible to show that only the subset $\boldsymbol{\rho}_* = (\rho_{p,q}\cos\theta_0, \rho_{p,q}\sin\theta_0)$ which satisfy

$$\sum_{j=1}^{q}\cos\left(\theta_0 + 2\pi j\frac{p}{q}\right)\sin\left[\rho_{p,q}\sin\left(\theta_0+2\pi j\frac{p}{q}\right)\right] = 0 \tag{8}$$

are also fixed points of the qth iteration of the first-order expansion $\mathcal{M}_* = (\mathcal{M}_0 + K\mathcal{M}_1)^q$. If p/q is an irrational number ι, then the circle defined by

$$\rho_\iota = \beta^{-1}[(\alpha/2\pi\iota)^2 - 1]^{1/2}, \tag{9}$$

is the conditionally periodic surface of winding number ι. The KAM theorem states that for some finite K there will be an invariant curve under the mapping $\mathcal{M}$ that impedes stochastic diffusion across it and that as $K \to 0$ this curve will approach a circle of radius ρ_ι.

We now expand $\mathcal{M}$ in powers of K and β such that

$$\beta^2 \rho_n^2 \simeq K \simeq \epsilon \ll 1. \tag{10}$$

Then equations (4) and (5) reduce to

$$u_{n+1} = (u_n \cos\alpha + v_n \sin\alpha) + K \sin v_n \cos\alpha + \tfrac{1}{2}\alpha\beta^2\rho_n^2(u_n \sin\alpha - v_n \cos\alpha), \tag{11}$$

$$u_{n+1} = (-u_n \sin\alpha + v_n \cos\alpha) - K \sin v_n \sin\alpha + \tfrac{1}{2}\alpha\beta^2\rho_n^2(u_n \cos\alpha + v_n \sin\alpha), \tag{12}$$

i.e.,

$$\boldsymbol{\rho}_{n+1} = \mathcal{R}_\alpha \cdot \boldsymbol{\rho}_n + K\mathbf{S}(\boldsymbol{\rho}_n) + \tfrac{1}{2}\alpha\beta^2 |\boldsymbol{\rho}_n|^2 \boldsymbol{\mathcal{R}}_{\alpha-\pi/2} \cdot \boldsymbol{\rho}_n, \tag{13}$$

where $\mathcal{R}_\alpha$ is the rotation matrix through α, and $\mathbf{S} = (\mathcal{R} \times \hat{\mathbf{x}})\sin(\hat{\mathbf{y}} \cdot \boldsymbol{\rho})$.

A considerable simplification in analyzing (13) occurs for the choice of $\alpha = 2\pi/4$, as noticed by Zaslavskii *et al.*[2] With this choice we readily obtain for the fourth iteration of $\mathcal{M}$

$$u_{n+4} = u_n + 2K \sin v_n - \pi\beta^2(u_n^2 + v_n^2)v_n, \tag{14}$$

$$v_{n+4} = v_n - 2K \sin u_n + \pi\beta^2(u_n^2 + v_n^2)u_n, \tag{15}$$

to first order in K and $\beta^2\rho^2$. Motion arising from a continuous-time Hamiltonian, H_4, will produce an identical mapping when it is integrated over periods of $4T$; this Hamiltonian is given by

$$H_4(u,v) = -\Omega_4\{[\cos v + \cos u + \Gamma(u^2+v^2)^2] + [\cos u + \Gamma(u^2+v^2)^2]\textstyle\sum_{j\neq 0}\cos(\frac{1}{4}j\Delta\omega t)\}, \tag{16}$$

with $\Omega_4 = K/2T$, $\Gamma = \pi\beta^2/8K$. The same result can be obtained through transformations on a Hamiltonian governing (1) and (2). The long time scale for the system evolution is $2\pi\Omega_4$ which is much greater than the periods of the time-dependent part of the Hamiltonian since $\Omega_4 \ll \Delta\omega$ for $K \ll 1$. Thus the time-averaged Hamiltonian is

$$\bar{H}_4(u,v) = -\Omega_4[\cos u + \cos v + \Gamma(u^2+v^2)^2] = \text{const}, \tag{17}$$

on the time scale of $2\pi/\Omega 4$.

The orbits of the particles obtained by (17) agree very well with those obtained by iteration of the full mapping. At small ρ, $\Gamma(u^2+v^2)^2$ is negligible and (17) reverts to the nonrelativistic case discussed in Ref. 2. For large ρ, i.e., high-particle energy, the $\Gamma(u^2+v^2)^2$ term dominates, making orbits which roughly circle the origin at constant radius.

The fixed points of (17), (u_0, v_0), are given by

$$\sin v_0 = 4\Gamma v_0(u_0^2 + v_0^2), \tag{18}$$

$$\sin u_0 = 4\Gamma u_0(u_0^2 + v_0^2). \tag{19}$$

No solutions to (18) and (19) can be found if

$$|4\Gamma\rho^2 \max(u,v)| > 1. \tag{20}$$

Since the smallest value of $\max(u,v)$ is $\rho\sqrt{2}$ there can be no solutions of (18) and (19), i.e., fixed points of $\bar{H}_4$, for particle energy γ above

$$\gamma_c = (1+\beta^2\rho_c^2)^{1/2} = [1 + 2(K\beta/\pi)^{2/3}]^{1/2}$$
$$= [1 + (4\sqrt{2}\omega_T^2/kc\Delta\omega)^{2/3}]^{1/2}, \tag{21}$$

for $\alpha = 2\pi/4$, where $\omega_T = (qEk/m)^{1/2}$ is the trapping frequency. Thus the nonrelativistic features of the map $\mathcal{M}$ defined by (4) and (5) for $\alpha = 2\pi/4$, i.e., the stochastic web, etc., are observed only for $\gamma < \gamma_c$. For $\gamma > \gamma_c$, $\rho > \rho_c$, KAM surfaces exist which are eventually destroyed as K gets larger and larger by the formation of islands, etc. For general $\alpha = 2\pi p/q$, the general expression for ρ_c in (21) must be replaced by $\rho_c = (2K/\alpha\beta^2)^{1/3}$. These predictions are confirmed over a range of K and β when the exact mapping, (4) through (6), is used.

Half of the fixed points given by (18) and (19) are hyperbolic (unstable) fixed points. Symmetries (18) and (19) allow us to group the fixed points into sets of four or eight. Points from a given set will have the same energy and stability properties by corresponding symmetries in (17). Two separatrix orbits will emanate from each hyperbolic point; these must subsequently terminate at a hyperbolic point from the same set since in general no other set will be at the correct energy. Thus each set of four or eight hyperbolic points is interconnected by a set of twice as many separatrices. Since separatrices of distinct energies cannot cross each other the sets must nest one inside the next (see Fig. 1). In this way Zaslavskii *et al.*'s infinite separatrix web[2] fractures into an infinity of distinct, nonintersecting separatrix families.

The time-dependent terms in the Hamiltonian (16) will perturb the separatrix motion and form the well known "stochastic layer" along the separatrices[7] (see

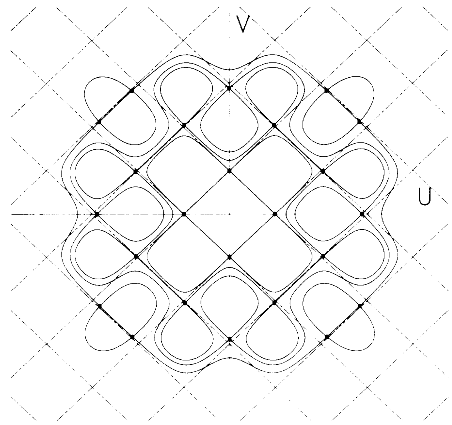

FIG. 1. Solid lines show four sets of separatrices of the averaged Hamiltonian (17) with $\Gamma = 1.3\times 10^{-4}$. Dots at the intersections represent the hyperbolic points. The dashed lines are the separatrices for the nonrelativistic Hamiltonian ($\Gamma = 0$).

Fig. 2). Zaslavskii *et al.* estimated the thickness of this layer to be $\delta(K) \sim \exp(-\pi^2/2K)$; since this is a local property of the map we expect this to hold for the relativistic map as long as $\beta^2\rho^2 \ll 1$. It should therefore be possible for neighboring separatrices to connect via layer overlap at points where they approach each other very closely. This overlap allows the distinct families of separatrices in some region to "fuse" together into a connected stochastic web on which Arnol'd diffusion can occur.

For $\Gamma\rho^3 \ll 1$ the solutions to (18) and (19) are, to first order, $\boldsymbol{\rho}_0 = (u_0, v_0) = (\pi i, \pi j)$, and the fixed point will be hyperbolic if $i+j$ is odd. Using (17), we can find the energy difference between neighboring sets of hyperbolic points to be $\Delta\overline{H}_4 = \overline{H}_4(\boldsymbol{\rho}_0'') - \overline{H}_4(\boldsymbol{\rho}_0') \simeq -\Omega_4\Gamma[(\rho_0'')^4 - (\rho_0')^4]$, where $\boldsymbol{\rho}_0''$ is a hyperbolic point from the outer set and $\boldsymbol{\rho}_0'$ from the inner set. The closest approach will come at that point on the separatrix where the gradient of the Hamiltonian (17) is a maximum. Since the separatrix can be given to first order by the condition[2] $\cos u + \cos v = 0$, or $\sin u = \pm \sin v$, the magnitude of the gradient is $|\nabla\overline{H}_4| \simeq \sqrt{2}\Omega_4 |\sin u| \le \sqrt{2}\Omega_4$. The separation, d, can then be estimated by

$$d \simeq |\Delta\overline{H}_4| / |\nabla\overline{H}_4| \simeq (\pi\beta^2/8\sqrt{2}K)[(\rho_0'')^4 - (\rho_0')^4]. \tag{22}$$

For a finite K we can decrease β, thereby decreasing the separation d, until it is less than the sum of the layer thicknesses, thus causing an overlap. For $\rho_0' \simeq \rho_0'' \gg \pi$ we can write

$$(\rho_0'')^4 - (\rho_0')^4 \simeq 4(\rho_0'')^3(\rho_0'' - \rho_0') \le 4\sqrt{2}\pi(\rho_0'')^3. \tag{23}$$

The upper bound comes from the fact that every set of hyperbolic points has a neighbor within $\sqrt{2}\pi$. The substitution of this into (22) gives the upper bound on d:

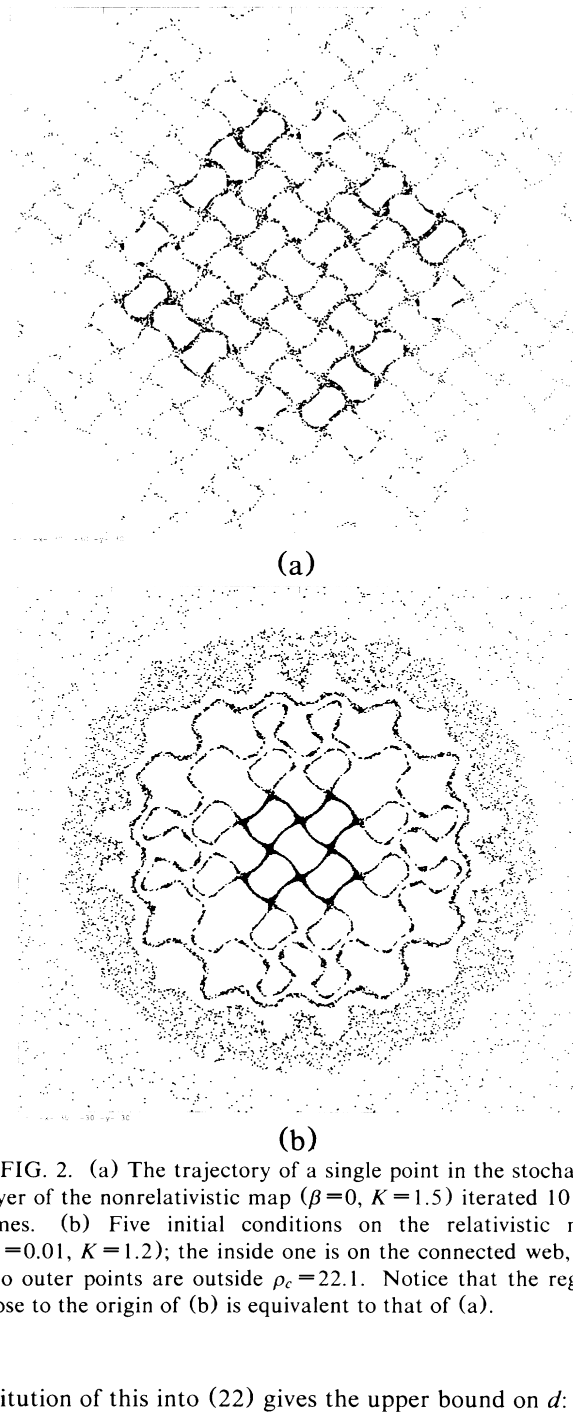

FIG. 2. (a) The trajectory of a single point in the stochastic layer of the nonrelativistic map ($\beta = 0$, $K = 1.5$) iterated 10000 times. (b) Five initial conditions on the relativistic map ($\beta = 0.01$, $K = 1.2$); the inside one is on the connected web, the two outer points are outside $\rho_c = 22.1$. Notice that the region close to the origin of (b) is equivalent to that of (a).

$$d \le (\pi^2\beta^2/2K)\rho_0^3 = \sqrt{2}\pi(\rho_0/\rho_c)^3. \tag{24}$$

If we state the overlap criterion in the form $d = 2\delta(K)$, where $\delta(K)$ is the layer thickness, then (23) implies that all pairs of separatrix sets inside the radius

$$\rho_d = [(\sqrt{2}/\pi)\delta(K)]^{1/3}\rho_c \sim (\sqrt{2}/\pi)^{1/3}\exp(-\pi^2/6K)\rho_c \tag{25}$$

will overlap. Since (23) is a very conservative upper bound we can expect layer overlap at radii many times ρ_d, but in light of (21) never beyond ρ_c.

The Arnol'd diffusion in the $1\frac{1}{2}$-dimensional Hamiltonian system which Zaslavskii *et al.*[2] reported is a direct consequence of the independence of the cyclotron frequency with respect to energy. It has been shown here that the introduction of a slight dependence due to relativistic effects causes the Arnol'd diffusion to be restricted to low-energy regimes of phase space. At a certain critical energy γ_c the energy (or action) dependence of the frequency becomes great enough to admit the application of the KAM theorem. Far below this energy the perturbing effects of the wave work to obscure any slight variation in frequency and Arnol'd diffusion can occur as shown by Zaslavskii *et al.*[2] However, a finite-amplitude perturbation is necessary to implement this mechanism; thus there is no diffusion for an arbitrarily small wave.

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