Simple Multifractal Cascade Model for Fully Developed Turbulence

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A simple model is presented for the energy-cascading process in the inertial range that fits remarkably well the entire spectrum of scaling exponents for the dissipation field in fully developed turbulence. The scheme is a special case of weighted curdling and its one-dimensional version is a simple generalized two-scale Cantor set with equal scales but unequal weights (with ratio $\simeq \frac{7}{3}$). This set displays all the measured multifractal properties of one-dimensional sections of the dissipation field.

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The statistical and scaling properties of the field of turbulent energy dissipation have been the subject of a number of studies,[1-5] and their proper description and modeling are vital for the understanding of fully developed turbulence. To explain the intermittent character of the rate of dissipation ϵ of turbulent kinetic energy, Kolmogorov[3] formulated his third hypothesis invoking some statistical independence in the cascading process, which led to the log-normal model for ϵ. This model presents internal inconsistencies and was later shown to be only one of many possibilities.[1,4,5] Mandelbrot[1] introduced a fractal model for ϵ, and distinguished between absolute and weighted curdling. The former became well known as the β model,[6] where the flux of energy is transferred to only a fixed fraction β of the eddies of smaller scales. This model, as well as the log-normal one, are contradicted by more recent experiments[7] concerning scaling exponents of velocity structure functions of high order. To correct this, Frisch and Parisi[8] proposed the multifractal model, where singularities of ϵ of different strengths are distributed on interwoven sets of different fractal dimension. We have shown elsewhere[2] that this description is consistent with experiments and that it provides a unifying framework for describing a variety of observations in fully developed turbulence. Benzi *et al.*[9] showed that multifractal distributions can be obtained if β is made a random variable in the β model. Another possibility is the weighted curdling introduced by Mandelbrot.[1] We will show that the simplest case of nonabsolute curdling describes (at a certain level of description) amazingly well the observed multifractal behavior of ϵ, and argue that from a metric point of view, fully developed turbulence and the proposed model (p model) are the same within experimental accuracy. As a detailed test of the p model, we compare its predictions with the entire spectrum of generalized dimensions,[10] and (equivalently) the singularity spectrum (the so-called f-α curve[11]), both of which have been measured by us[2] for the energy-dissipation field in several turbulent flows (see Figs. 1 and 2).

Let E_r be the dissipation that occurs in a domain of size r. It is convenient to think of E_r as the flux of kinetic energy[6] from eddies of size r to eddies of smaller size—this flux being actually dissipation when these eddies are of the order of the Kolmogorov length scale η. This is the classical view of the eddy cascade in the inertial range, and in what follows we will work within that framework, even though we know that it is an oversimplification.

Suppose that an eddy size r breaks down into 2^d eddies of equal size $r/2$, d being the dimensionality of the space we are analyzing. Furthermore, suppose that the flux of energy to these smaller eddies proceeds unequally. The simplest nontrivial choice is a fraction p_1 distributed equally among one half of the 2^d new eddies, and a fraction $p_2 = 1 - p_1$ distributed similarly among the other

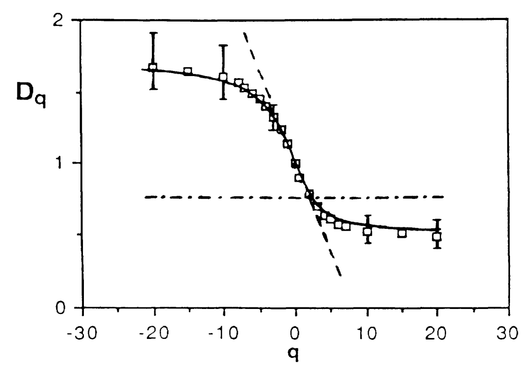

FIG. 1. Generalized dimensions for one-dimensional sections through the dissipation field in several fully developed turbulent flows (grid turbulence, wake of a circular cylinder, boundary layer, atmospheric turbulence). The experimental details can be found in Ref. 2 (for a discussion of the validity of the results at $q < 0$, see, especially, Appendix C). The symbols correspond to the experimental mean (which is independent of the type of flow within experimental accuracy) and the continuous curve to the present model with $p_1 = 0.7$. The error bars (also essentially independent of the type of flow) correspond to the variability of the D_q observed at different regions of the one-dimensional sections (these regions being much larger than the integral length scale L), and/or in different realizations of the experiment. The statistics of the measured D_q's are such that their mean values are stable and representative. The dashed line corresponds to the log-normal model, and the dot-dashed line to the β model, both for $\mu = 0.25$.

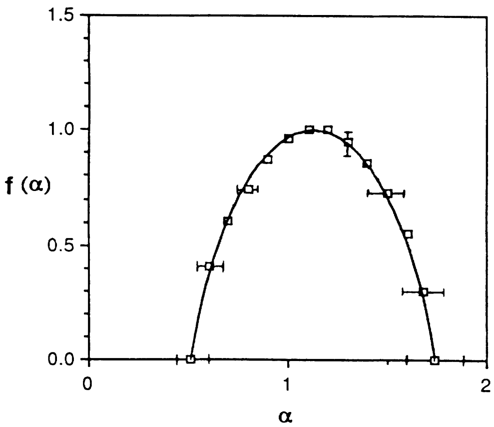

FIG. 2. The multifractal spectrum, or the f-α curve, corresponding to the D_q curve of Fig. 1. Symbols and the continuous curve as in Fig. 1.

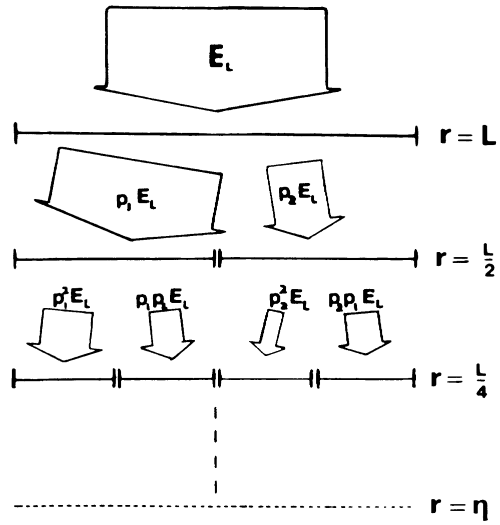

FIG. 3. One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into nonequal fractions p_1 and p_2. This cascade terminates when the eddies are of the size of the Kolmogorov scale, η.

half. This process is repeated with fixed p_1 over and over, until one reaches eddies of order η. A one-dimensional section of the resulting multifractal will be equivalent to consideration of $d=1$ (see Fig. 3). Note that this multifractal distribution corresponds to a generalized two-scale Cantor set[11] with $l_1=l_2=\frac{1}{2}$. For now, we will focus on the case $d=1$ since our experimental results were obtained for linear intersections of the flow, but we will indicate later how to extend the results to $d=3$.

The generalized dimensions[10] D_q of the resulting distribution at the nth stage of the cascade can be calculated analytically for this case with the following identity:

$$\sum E_r^q = E_L^q (r/L)^{(q-1)D_q}, \tag{1}$$

where the sum is taken over all eddies at the nth stage, L is the size of the initial eddy (corresponding roughly to the integral length scale in turbulence), and E_L is the total dissipation in a domain of size L. At the nth stage of the cascade, the eddy size will be $r=L(\frac{1}{2})^n$ and there will be $\binom{n}{m}$ eddies where $E_r = p_1^{n-m} p_2^m E_L$ for all integers m between 0 and n. The sum over all eddies can therefore be recognized as the binomial expression

$$E_L^q (p_1^q + p_2^q)^n = E_L^q (\tfrac{1}{2})^{n(q-1)D_q}. \tag{2}$$

It follows that

$$D_q = \log_2 \{p_1^q + p_2^q\}^{1/(1-q)}. \tag{3}$$

Taking the limits for $q=\pm\infty$ and using the result for the intermittency exponent,[2] $\mu = -2dD_q/dq\,|_{q=0}$, we have (supposing that $p_1 > \frac{1}{2}$)

$$D_\infty = \log_2 p_1^{-1}, \tag{4}$$

$$D_{-\infty} = \log_2 p_2^{-1}, \tag{5}$$

and

$$\mu = \log_2 (4p_1p_2)^{-1}. \tag{6}$$

We can now use (4) to obtain p_1 given our experimental result for D_∞. Using[2] $D_\infty = 0.51$, we get $p_1 = 0.7$. With $p_2 = 1 - p_1$, we get independently from (5) and (6) that $D_{-\infty} = 1.74$ and $\mu = 0.25$, values that are in very good agreement with our experimental results.[2] In Fig. 1, we show a comparison between the measured D_q curve and Eq. (3) for $p_1 = 0.7$. The agreement is remarkable. Note that $D_0 = 1$ in our model, implying that every eddy that is generated receives some flux of energy. It is worth mentioning that if we had used nonequal eddies, for example $l_1 = \frac{1}{3}$ and $l_2 = \frac{2}{3}$, no choice of p_1 would have been satisfactory. (Given the level of accuracy of the experiments, it does not seem justified to improve the fit by changing p_1 and l_1 slightly around 0.7 and $\frac{1}{2}$.) If we were to include the possibility that eddies in the inertial range (with $r > \eta$) also dissipate some energy directly, we can consider $p_1 + p_2 < 1$, which gives a second free parameter p_2. It again turns out that the only reasonable fit to the D_q curve occurs with $p_1 = 0.7$ and $p_2 = 0.3$, providing a further illustration of the robustness of the present model.

We can also get an analytic expression for the f-α curve[2,11] using results for a general two-scale Cantor set obtained by Halsey *et al.*[11]:

$$\alpha = \frac{\log_2 p_1 + (n/m - 1)\log_2 p_2}{\log_2 l_1 + (n/m - 1)\log_2 l_2}, \tag{7}$$

$$f(\alpha) = \frac{(n/m - 1)\log_2(n/m - 1) - (n/m)\log_2(n/m)}{\log_2 l_1 + (n/m - 1)\log_2 l_2}. \tag{8}$$

Eliminating n/m and using $l_1 = l_2 = \frac{1}{2}$, we get an explicit expression for $f(\alpha)$ as a function of α, which is depicted as the continuous curve in Fig. 2, along with the experimental data.[2] Again, the agreement is excellent.

We can also construct an artificial signal of ϵ from the p model by starting with an interval of length L and height equal to the mean dissipation ϵ_L. (This quantity, being the flux of energy per unit volume that is injected at the large scales in the cascading process, can be estimated with $\epsilon_L = u'^3/L$, where u' is the root-mean-square value of the velocity.) Then we divide this interval into two segments of length $L/2$ and assign to the segments heights of $2p_1\epsilon_L$ and $2p_2\epsilon_L$ with $p_1 = 1 - p_2 = 0.7$. This is repeated for each remaining segment, selecting the left or right position for p_1 at random. Figures 4(a)–4(c) show the first, fifth, and twelfth stages of the construction. Different realizations can be obtained by changing the seed in the random selection of the right or left segment for p_1. Figure 4(d) is an actual signal of $\epsilon \sim \nu U^{-2}(\partial u/\partial t)^2$ obtained from experiments at $R_\lambda = 200$. This corresponds approximately to the twelfth stage of construction if we use

$$R_\lambda^{-3/2} = \eta/L \simeq (\tfrac{1}{2})^n. \tag{9}$$

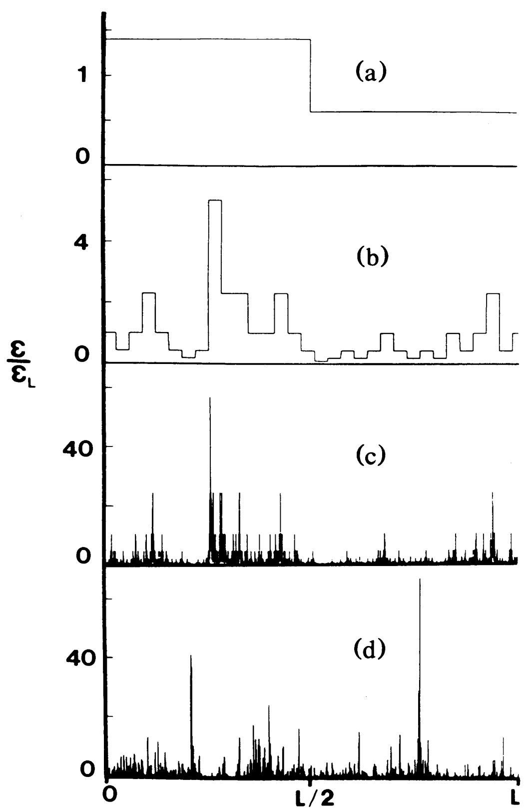

FIG. 4. Different stages during the construction of the proposed p model of the dissipation field [(a) first stage, (b) fifth stage, (c) twelfth stage], and (d) an experimental signal of ϵ, corresponding to the twelfth stage of construction (see text).

To exemplify further the appropriateness of this model, we calculate the second moment of the generated signals at different stages. The second moment of ϵ can be regarded as the flatness factor (that is the normalized fourth moment) K of velocity derivatives at different Reynolds numbers, for which there is a lot of experimental information available.[12] Earlier, we showed[2] that $K \sim R_\lambda^{3/2(1-D_2)}$, and since we can obtain D_2 from Eq. (3) for $q = 2$, we can write

$$K \sim R_\lambda^{3/2\{1+\log_2(p_1^2+p_2^2)\}}. \tag{10}$$

Figure 5 bears out this relation for $p_1 = 0.7$ and the measured second moment of the generated signals at different stages. It is seen that the power-law exponents of the p model are consistent with the data. Since we recognize that (9) is only qualitative, the agreement of the model with the data can be improved (see Fig. 5) by writing (empirically) that

$$\eta/L = 3(\tfrac{1}{2})^n. \tag{11}$$

For $d = 3$ we obtain for the generalized dimensions $D_{q,3}$ embedded in three-dimensional space

$$\sum (E_r/E_L)^q = [4(p_1/4)^q + 4(p_2/4)^q]^n = (\tfrac{1}{2})^{n(q-1)D_{q,3}}. \tag{12}$$

Therefore, we have

$$D_{q,3} = 2 + \log_2\{p_1^q + p_2^q\}^{1/(1-q)} = D_q + 2, \tag{13}$$

as expected for isotropic multifractals (see Appendix A of Ref. 2).

We conclude that from a "thermodynamic" point of view,[13] both the dissipation field of turbulence and this process of asymmetric energy flux to smaller eddies belong to the same class of scaling processes (within exper-

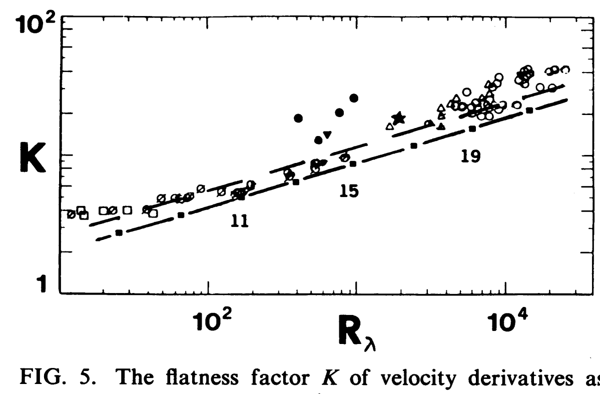

FIG. 5. The flatness factor K of velocity derivatives as a function of Reynolds number (based on the so-called Taylor microscale). Different symbols correspond to experimental data in different flows as collected by Van Atta and Antonia (Ref. 12). The line through black squares corresponds to the prediction from the p model (with $p_1 = 0.7$) at different stages n (indicated by the numbers). The dashed line corresponds to Eq. (10) using (11) for η/L.

imental accuracy). The latter can therefore be regarded as a convenient model for ϵ, especially if one is interested in modeling correctly the scaling properties of ϵ. We know of no other simple model that incorporates satisfactorily the great variety of observed scaling exponents (for a relation between those commonly encountered measures in the turbulence literature and the D_q curve, see Meneveau and Sreenivasan[2]).

Since the D_q curve and the f-α curve correspond to a "thermodynamic" description of the scaling process which does not contain the entire scaling information,[13] we cannot conclude that the eddy breakdown is governed exactly by a single two-scale Cantor set with $p_1=0.7$. However, the excellent agreement of the model with experiments, and its simplicity which renders redundant the need for a more complicated mechanism, both strongly indicate that the two processes are related. To find possible dynamical reasons for this asymmetric breakdown of eddies, possibly relating to the symmetry-breaking properties of the Euler equations, as well as to explore further variants of this model, are interesting and promising tasks for the future.

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