Chiral-Symmetry-Breaking Phase Transition in Lattice Gauge-Higgs Theories with Fermions

I-Hsiu Lee
Physics Department, Brookhaven National Laboratory, Upton, New York 11973

and

Robert E. Shrock
Institute for Theoretical Physics, State University of New York, Stony Brook, Stony Brook, New York 11794
(Received 30 March 1987)

We present an analytic derivation of a chiral-symmetry-breaking transition in (zero temperature) U(1) and Z_N lattice gauge-Higgs theories with fermions. Several remarkable properties of this transition are discussed. Implications for confining models of weak interactions are noted.

PACS numbers: 11.15.Ha, 11.30.Rd, 12.15.−y

One of the important features of quantum chromodynamics is spontaneous chiral-symmetry breaking (SχSB). One may ask whether this would occur in a theory in which the gauge symmetry is spontaneously "broken." Since SχSB is a nonperturbative phenomenon, it is advantageous to use a lattice formulation to study this question. Because the confinement and Higgs phases are connected in a lattice gauge theory with Higgs fields in the fundamental representation[1] one might expect that when fermions are added, these two phases would again be analytically connected to each other. Since $\langle\bar{\psi}\psi\rangle$ is nonzero in the confinement phase of a lattice gauge theory with fermions, it would follow that when such Higgs fields are added, this chiral condensate would also be nonzero for all finite β_h (with $0 \le \beta_g < \infty$) throughout the Higgs phase.

However, this naive expectation was not borne out by a recent Monte Carlo study[2] of a particular model [four-dimensional SU(2) with $I=\frac{1}{2}$ scalars and $I=\frac{1}{2}$, 1 fermions, in the quenched approximation], which yielded evidence for chiral-symmetry restoration for β_h larger than a critical value, $\beta_{h,c}(\beta_g)$. By standard arguments, $\langle\bar{\psi}\psi\rangle$ must vanish nonanalytically across the curve defined by $\beta_{h,c}(\beta_g)$, justifying the term "chiral phase transition."

In the present Letter we present the first analytic study of lattice gauge theories with both Higgs and fermion fields and demonstrate that the chiral phase transition occurs in a wide class of such theories. We consider a generic theory based on a gauge group $G=\mathrm{U}(1)$ or Z_N (at zero temperature), formulated on a (hyper)cubic d-dimensional lattice, defined in standard notation by

$$Z=\int\prod_{n,\mu} d\chi_n\, d\bar{\chi}_n\, d\phi_n\, d\phi_n^\dagger dU_{n,\mu} e^{-S}, \tag{1a}$$

where the integral is understood as a sum for discrete G and $S=S_g+S_h+S_f$, with[3]

$$S_g=\beta_g \sum_{\rm plaq}[1-\mathrm{Re}(U_{\rm plaq})], \tag{1b}$$

$$S_h=-\beta_h\sum_{n,\mu}[\phi_n^\dagger(U_{n,\mu})^{q_h}\phi_{n+e_\mu}+\mathrm{H.c.}], \tag{1c}$$

$$S_f=\tfrac{1}{2}\sum_n\bar{\chi}_n\sum_\mu \eta_{n,\mu}[(U_{n,\mu})^{q_f}\chi_{n+e_\mu}-(U^\dagger_{n-e_\mu,\mu})^{q_f}\chi_{n-e_\mu}]+m\sum_n\bar{\chi}_n\chi_n, \tag{1d}$$

with $\beta_g=4/g^2$. The lattice Higgs fields are taken to have fixed length; as is well known, this does not, in general, imply that the continuum fields are of fixed length. With no loss of generality, we take $\beta_h \ge 0$. The χ_n and $\bar{\chi}_n$ are one-component, anticommuting, staggered fermion fields[4] assigned to each lattice site n; the $\eta_{n,\mu}=\{1$ for $\mu=1$; $(-1)^{n_1+\cdots+n_{\mu-1}}$ for $\mu=2,\ldots,d\}$ are associated factors arising from Dirac matrices.[4,5] It is advantageous to use staggered fermions in studies of chiral symmetry since they retain a continuous remnant of this symmetry on the lattice. For $G=\mathrm{U}(1)$, q_f and q_h denote the fermion and Higgs charges; for Z_N, $q_f=q_h=1$. Since we wish to show the existence of a new nonanalytic boundary between the confinement and Higgs phases due to the presence of fermions, we shall (unless otherwise stated) take $q_h=1$; for $q_h \ge 2$, there would already be such a boundary even in the absence of fermions.[6] The mass term is included for generality. For technical reasons, we consider the strong-gauge-coupling limit, $\beta_g=0$.

Usually, one begins by integrating out the fermions. This has the advantage that thereafter one does not deal with anticommuting variables, but the disadvantage that it yields a nonlocal fermion determinant. We take a different approach, performing the integration over Higgs and gauge fields first. This yields a formulation of the theory entirely in terms of a fermionic path integral,

$Z = J_1^{N_l} \int \prod_n d\chi_n d\bar{\chi}_n e^{-S'}$, where N_l is the number of links in the lattice and

$$S' = -\sum_{n,\mu} [\tfrac{1}{4}(1-r^2)\bar{\chi}_n\chi_n\bar{\chi}_{n+e_\mu}\chi_{n+e_\mu} - \tfrac{1}{2} r\eta_{n,\mu}(\bar{\chi}_n\chi_{n+e_\mu} - \bar{\chi}_{n+e_\mu}\chi_n)] + m\sum_n \bar{\chi}_n\chi_n, \tag{2a}$$

with $r = J_U/J_1$ and

$$J_1 = \int dU\, e^{2\beta_h \mathrm{Re}(U^{q_h})}, \tag{2b}$$

$$J_U = \int dU\, e^{2\beta_h \mathrm{Re}(U^{q_h})} U^{q_f}. \tag{2c}$$

All dependence on the group G enters only through $r(\beta_h)$, which is an analytic function that increases monotonically from $r(0)=0$ to $r(\infty)=1$. [Hence, the inverse function $\beta_h(r)$ is well defined.] Explicitly, $J_1 = I_0(2\beta_h)$, $J_U = I_{q_f}(2\beta_h)$ for U(1) (and $q_h = 1$); $J_1 = \cosh(2\beta_h)$, $J_U = \sinh(2\beta_h)$ for Z_2; and so forth for Z_N with $N > 2$. The first and third terms in S' are explicitly gauge invariant, and the only terms in the expansion of the exponential of the second term which survive the fermionic integration are closed loops C of the form $\prod_{n\in C}\bar{\chi}_n\chi_n$ and hence are also gauge invariant.

Next, we use a mean-field-theory (MFT) technique, replacing $\sum_{n,\mu}\bar{\chi}_n\chi_n\bar{\chi}_{n+e_\mu}\chi_{n+e_\mu}$ in (2a) by $2d\langle\bar{\chi}\chi\rangle\sum_n\bar{\chi}_n\chi_n$. At $\beta_h = 0$, where the Higgs fields play no role and the theory reduces to a pure gauge-fermion theory (with $\beta_g = 0$), $\langle\bar{\chi}\chi\rangle$ is certainly nonzero[7]; the question is for what range of $\beta_h > 0$ does this continue to hold? Our MFT method still leaves us with an integral involving an infinite number of coupled degrees of freedom, in contrast to the use of MFT in a typical statistical-mechanical spin model, which essentially reduces the partition function to a single-site problem. However, because the action is rendered quadratic in fermion fields, we can perform the infinite number of coupled integrations exactly. Calculating $\langle\bar{\chi}\chi\rangle$, we obtain the consistency equation

$$\langle\bar{\chi}\chi\rangle = \frac{d}{2}(1-r^2)\langle\bar{\chi}\chi\rangle \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d} \left\{ \left[\frac{d}{2}(1-r^2)\langle\bar{\chi}\chi\rangle\right]^2 + r^2 \sum_{\mu=1}^{d} \sin^2 p_\mu \right\}^{-1}. \tag{3}$$

As usual with MFT consistency equations, (3) always has the solution $\langle\bar{\chi}\chi\rangle = 0$; however, it is easily checked that if (3) does have a solution for nonzero $\langle\bar{\chi}\chi\rangle$, this minimizes the free energy and hence is the physical solution. Next, observe that for $\beta_h = 0$, the solution of (3) is $\langle\bar{\chi}\chi\rangle = (2/d)^{1/2}$.[8] We thus define

$$M \equiv \frac{\langle\bar{\chi}\chi\rangle}{\langle\bar{\chi}\chi\rangle|_{r=0}}. \tag{4}$$

We have solved (3) numerically for the physical case $d=4$ and, for comparison, $d=3$; the results are shown in Fig. 1. With use of these results and the properties of r, it follows that $\langle\bar{\chi}\chi\rangle$ is a nonincreasing function of β_h. Further, for r greater than the critical value r_c depending on d, or equivalently, $\beta_h > \beta_{h,c} = \beta_h(r_c)$, (3) has only the chirally symmetric solution $\langle\bar{\chi}\chi\rangle = 0$. Quite generally, given that $\langle\bar{\chi}\chi\rangle$ vanishes everywhere along the segment $\beta_{h,c} < \beta_h \le \infty$, but is nonzero for $0 \le \beta_h < \beta_{h,c}$, it follows that $\langle\bar{\chi}\chi\rangle$ must be nonanalytic at $\beta_{h,c}$. These results answer the basic question posed above and show that as β_h increases from 0 to ∞, $\langle\bar{\chi}\chi\rangle$ vanishes nonanalytically at a finite value of β_h rather than smoothly decreasing to 0 at $\beta_h = \infty$. Our MFT method predicts the chiral phase transition to be continuous.

This chiral phase transition has a remarkable property which distinguishes it from phase transitions in all other statistical-mechanical models of which we are aware: in the latter, the (again, global) symmetry is realized explicitly for $0 \le \beta < \beta_c$ (where in that context, $\beta \equiv (k_B T)^{-1}$ and β_c may $=\infty$), and possibly spontaneously broken for $\beta > \beta_c$. Here it is the exact opposite: the chiral symmetry is spontaneously broken for $0 \le \beta_h < \beta_{h,c}$ and restored (for $q_h = 1$)[6] for $\beta_h > \beta_{h,c}$.

These results provide analytic insight into the numerical findings of Ref. 2. For technical simplicity, we have restricted ourselves to Abelian G, but in the strong-gauge-coupling limit this should not be an important difference.

We obtain for r_c^2 the values 0.585 ± 0.008 and 0.551 ± 0.005 for $d=3$ and 4, respectively. These values illustrate a general feature, which is that r_c^2 is a decreasing

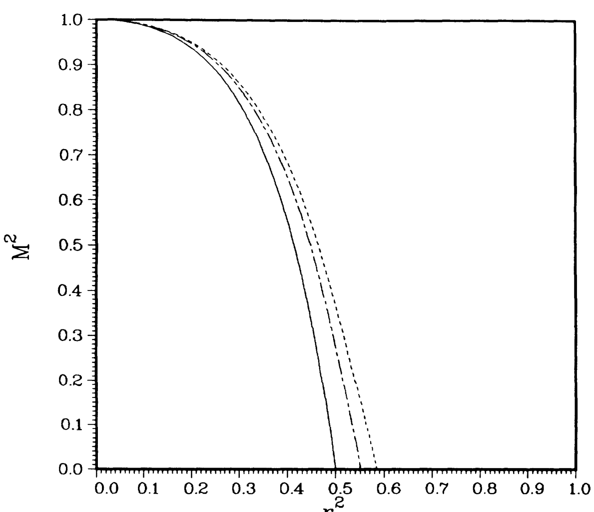

FIG. 1. Plot of solution to Eq. (3). Dashed, dot-dashed, and solid curves refer to $d=3$, 4, and ∞, respectively.

function of d [this property is seen analytically from Eq. (8) below]. Second, since $I_{q_f}(2\beta_h)/I_0(2\beta_h)$ is a decreasing function of q_f for fixed β_h, it follows that for U(1), $\beta_{h,c}$ is an increasing function of q_f. This makes sense physically, since one ascribes the SχSB to the gauge-fermion interaction, and this is strengthened by increasing q_f, so that the SχSB phase is expected to extend further up into the phase diagram, as a function of β_h. Solving for $\beta_{h,c}$ for various G yields the results in Table I. We find that $\beta_{h,c;Z_N}$ is an increasing function of N and (as a corollary) $\beta_{h,c;Z_N} < \beta_{h,c;\mathrm{U}(1)}$. Because our MFT method is an approximation (which neglects fluctuations), its predictions for $\beta_{h,c}$ for a given G and d will, of course, differ somewhat from the true values; experience with typical spin models suggests that for $d=4$, the difference would be about 10%–20%. However, for the dimensionalities of interest here, we expect the qualitative predictions of MFT to be reliable.

The limit of large d gives further insight. Here we are able to obtain the exact solution of (3) (and hence of the full theory, since MFT should be exact in the limit $d \to \infty$):

$$M^2 = (1-2r^2)/(1-r^2)^2, \quad \text{for } d \to \infty \tag{5}$$

(see Fig. 1) which yields

$$\lim_{d\to\infty} r_c^2 = \tfrac{1}{2}. \tag{6}$$

The resultant values of $\beta_{h,c}$ are listed in Table I. In particular, for $d \to \infty$,

$$\beta_{h,c;Z_2} = \tfrac{1}{2}\beta_{h,c;Z_4} = \ln(1+\sqrt{2}), \tag{7a}$$

$$\beta_{h,c;Z_3} = \tfrac{1}{3}\ln(4+3\sqrt{2}). \tag{7b}$$

Evidently, the solution for $d=4$ is already rather close to that for $d=\infty$. These result are quite different from those in all statistical-mechanical models of which we are aware, in that for the latter, as $d\to\infty$, the critical point $\beta_c \to 0$ (for a typical spin model, as $d\to\infty$, $\beta_c \to \beta_{c,\mathrm{MFT}} = \mathrm{const}/2d \to 0$). In contrast, here the critical point approaches a finite constant in this limit. Secondly, we know of no statistical-mechanical model where the order parameter (at a given β where it is nonzero) goes to zero as $d\to\infty$, but here $\langle\bar\chi\chi\rangle$ does indeed vanish in this limit. From (5), it follows that at $d=\infty$, the critical exponent $\beta_{\exp}$ for the order parameter M has the usual MFT value, $\frac{1}{2}$; Fig. 1 suggests that this value also holds for finite d, again the usual MFT behavior.

We have calculated, by steepest descent methods, a $1/d$ expansion of (3). Define $\lambda \equiv 2r^{-2}(1-r^2)^2M^2$, and $\xi \equiv (1+\lambda/2)^{-1}$. Then (3) (after division by $\langle\bar\chi\chi\rangle$) becomes

$$1 = (r^{-2}-1)\xi\left[\sum_{k=0}^{6} c_k \xi^{2k}(2d)^{-k} + O(d^{-7})\right], \tag{8}$$

where $c_0=1$, $c_1=1$, $c_2=3$, $c_3=3(5-\xi^{-2})$, $c_4=15(7-3\xi^{-2})$, $c_5=5(189-126\xi^{-2}+8\xi^{-4})$, and $c_6=35(297-270\xi^{-2}+41\xi^{-4})$. We have solved (8) for r_c^2 and have obtained excellent agreement with the results from the numerical solution of (3).

Our work shows that when one incorporates fermions into gauge-Higgs lattice models with fundamental-representation Higgs a new nonanalytic boundary associated with a chiral-symmetry-breaking transition appears which was not present in the pure gauge-Higgs theory. This has important consequences for studies of electroweak interactions based on the full path integral, going beyond perturbation theory. In particular, our work has implications for the (continuum) confining model of weak interactions.[9-11] Although this model was originally partly motivated by the finding that the confinement and fundamental-representation Higgs phases are analytically connected, it was necessary to assume both that the underlying SU(2) confines and that chiral symmetry is not broken. (If the latter occurred, it would, e.g., violate electric charge conservation and lead to fermion masses of order 250 GeV.) However, we find that in the confinement phase (defined, as usual, as the set of points in the phase diagram which are analytically connected to the confinement phase of the pure gauge theory) chiral symmetry is broken, while in the Higgs phase (defined, again as usual, as the set of points which are analytically connected to the SSB phase of the global theory at $\beta_g=\infty$) the chiral symmetry is restored. This lattice result is relevant to the continuum theory since the latter is defined as a limit of the lattice theory as the lattice spacing goes to zero at a continuous phase transition, approached from within a given phase, and consequently, the properties of the continuum theory reflect the phase of the lattice theory in which this limit was taken.

We thank M. Creutz, E. Farhi, R. Jaffe, C. Rebbi, J. Shigemitsu, and J. Smit for their comments. One of us (I-H.L.) is partially supported by the U.S. Department of Energy under Contract No. DE-AC02-76CH00016. Another of us (R.E.S.) is partially supported by the National Science Foundation under Contract No. PHY-85-07627.

TABLE I. Values of $\beta_{h,c}$ for various G and d.

G	$d=3$	$d=4$	$d=\infty$
U(1), $q_f=1$	1.25 ± 0.02	1.15 ± 0.01	1.03
U(1), $q_f=2$	3.98 ± 0.10	3.60 ± 0.05	3.13
U(1), $q_f=3$	8.63 ± 0.22	7.78 ± 0.12	6.72
Z_2	0.504 ± 0.006	0.478 ± 0.004	0.441
Z_3	0.792 ± 0.009	0.755 ± 0.005	0.703
Z_4	1.01 ± 0.01	0.956 ± 0.008	0.881

[1]E. Fradkin and S. H. Shenker, Phys. Rev. D 19, 3682 (1979).

[2]I.-H. Lee and J. Shigemitsu, Phys. Lett. B **178**, 93 (1986).

[3]The notational convention for β_h in (1c) is as in Ref. 2, so that $(\beta_h) \equiv \frac{1}{2}(\beta_h)_{\rm alt}$, where the alternative convention is used, e.g., in R. E. Shrock, Phys. Lett. **162B**, 165 (1985), and Phys. Rev. Lett. **56**, 2124 (1986), and Nucl. Phys. **B267**, 301 (1986), and **B278**, 380 (1986), etc.

[4]T. Banks, L. Susskind, and J. Kogut, Phys. Rev. D **13**, 1043 (1977); L. Susskind, Phys. Rev. D **16**, 3031 (1977).

[5]N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981).

[6]Parenthetically, for U(1) and general q_h, we note the following results: The $J_i(q_f,q_h)$, $i=1$, U satisfy property $J_i(q_f,q_h)=J_i(k_{q_f},k_{q_h})$ if k is a nonzero integer. Further, for k a nonzero integer and nonzero q_f, $J_U(q_f,q_h=2k_{q_f})=0$. This implies that at $\beta_g=0$, for $q_h=2k_{q_f}\neq 0$, the bosonic and fermionic sectors decouple, i.e., the partition function, Z, factorizes: $Z=Z_hZ_f$, where $Z_h=J_1^{N_l}$ is the partition function for the pure gauge-Higgs theory, and Z_f, the fermionic factor, is independent of β_h. Hence, $\langle\bar{\chi}\chi\rangle$ is independent of β_h and, for this case, there is no chiral symmetry restoration transition. This makes good sense since, for $q_h\neq 0$, the entire axis $\beta_g=0$ lies within the confinement phase, and at $\beta_h=\infty$, rather than reducing to free fermions, the theory reduces to a pure Z_{2k} gauge-fermion model. These are exact results; from (3), we obtain, in our MFT approximation, $\langle\bar{\chi}\chi\rangle=(2/d)^{1/2}$.

[7]For an early review of SχSB in pure lattice gauge-fermion theory, see, e.g., J. Kogut, Rev. Mod. Phys. **55**, 775 (1983).

[8]This agrees with the MFT special case of the work on pure gauge-fermion theory at $\beta_g=0$ in Ref. 5 and by H. Kluberg-Stern, A. Morel, and B. Petersson, Phys. Lett. **114B**, 152 (1982), and Nucl. Phys. **B215**, 527 (1983); see also J.-M. Blairon *et al.*, Nucl. Phys. **B180**, 439 (1981) [SU(N)]; because the latter work uses a very different, $N\rightarrow\infty$ limit, it is not clear that it would obtain the same result, but, aside from an obvious group-theory factor due to the difference in G, it does.

[9]L. Abbot and E. Farhi, Phys. Lett. **101B**, 69 (1981), and Nucl. Phys. **B189**, 547 (1981).

[10]M. Claudson, E. Farhi, and R. L. Jaffe, Phys. Rev. D **34**, 873 (1986).

[11]Given that the $U(1)_Y$ is neglected in this model (Refs. 9,10), it is effectively vectorlike, since SU(2) has only real representations, and half of the fermions can be represented as their right-handed charge conjugates. Hence, although in Ref. 2 and the present work the fermions are coupled in a vectorial manner to the gauge field, the conclusions of these works should be applicable to the model of Refs. 9 and 10. (In Ref. 10, a SχSB transition is conjectured to occur.)