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Organization of Chaos

Gemunu H. Gunaratne[(1)] and Itamar Procaccia[(1,2)]
[(1)]The James Franck Institute, The University of Chicago, Chicago, Illinois 60637
[(2)]Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
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The organization, encoding, and hierarchical construction of a generic chaotic attractor is presented. A systematic calculation of the multifractal properties is accomplished, and an understanding of the spectrum of singularities is reported. In general we expect nontrivial phase transitions in the thermodynamic formalism of strange attractors.

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The quantitative understanding of the structure and properties of generic strange attractors is one of the remaining outstanding problems in the field of low-dimensional chaos.[1] To achieve such an understanding for typical attractors three basic ingredients are needed: (i) a hierarchical scheme to describe the set and its natural invariant measure with sets of increasing complexity which are, however, under control; (ii) an encoding which will give a unique address to all the points in these sets of increasing complexity; and (iii) a relationship between the scaling exponents that characterize the invariant measure and the properties of the points in the hierarchical scheme. All these ingredients were at the basis of the success of elucidating the nature of sets that live at the borderline of chaos.[2] It appears that now we have reached a position where we can attempt to achieve similar quantitative understanding of chaotic attractors that are embedded in two dimensions. The aim of this Letter is to report these results.

Consider a dissipative dynamical system $\mathbf{x}_{n+1} = \mathbf{M}_\mu(\mathbf{x}_n)$, where $\mathbf{x}_n \in R^2$, and μ is a set of parameters. The linearized map at the point $\mathbf{x}_n$ will be denoted J_n. Consider some values of the parameters μ where the orbit is confined to a nontrivial compact chaotic attractor. At the basis of our approach lies the realization that by looking at sets of longer and longer periodic orbits which belong to the closure of the attracting set (being actually dense on typical attractors) we can hierarchically approach the chaotic set.[3,4] Evidence for this conjecture for both generic (nonhyperbolic) and nongeneric (hyperbolic)[5] attractors was given recently. We know that the number of periodic points of periods of length m grows exponentially with m, the exponent being the topological entropy.[1] We can further estimate the probability of seeing a particular orbit of length m—this probability is inversely proportional to the positive Lyapunov number of the orbit.[6,7]

Looking at an m-periodic orbit $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{x}_1, \ldots, \mathbf{x}_m, \ldots$, we define the Lyapunov numbers as the eigenvalues of the matrix $J_m \cdot J_{m-1} \cdots J_2 \cdot J_1$, and write them as $\exp(\lambda_1^{(m)})$ and $\exp(\lambda_2^{(m)})$, where the convention will be that $\exp(\lambda_1^{(m)}) > 1$ and $\exp(\lambda_2^{(m)}) < 1$. The set of all points belonging to periodic orbits of period m will be denoted by fix$m$. The probability of a particular point $\mathbf{x} \in \mathrm{fix}m$ is then equal to

$$\frac{\exp[-\lambda_1^{(m)}(\mathbf{x})]}{\sum_{\mathbf{x}\in \mathrm{fix}m} \exp[-\lambda_1^{(m)}(\mathbf{x})]},$$

and an nth-order approximant to the invariant measure can be written as

$$\rho^{(m)}(\mathbf{y}) = \frac{\sum_{\mathbf{x}\in \mathrm{fix}m} \delta(\mathbf{y}-\mathbf{x}) \exp[-\lambda_1^{(m)}(\mathbf{x})]}{\sum_{\mathbf{x}\in \mathrm{fix}m} \exp[-\lambda_1^{(m)}(\mathbf{x})]}. \quad (1)$$

The way that we want to characterize the scaling properties of the invariant measure is via the $f(\alpha)$ formalism.[8] In this formalism we consider the natural invariant measure defined by a long chaotic time series $\{\mathbf{x}_i\}_{i=1}^N$, and define the natural measure $P_n(l_1,l_2)$ of a box of size $l_1 \times l_2$ centered about the nth point $\mathbf{x}_n$ according to [5,9]

$$P_n(l_1,l_2) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} E_n(\mathbf{x}_i - \mathbf{x}_n), \tag{2}$$

where $E_n(\mathbf{x}-\mathbf{x}_n)=1$ if $|\zeta| < l_1$ and $|\eta| < l_2$, and zero otherwise, with $\mathbf{x}-\mathbf{x}_n = \zeta\hat{\mathbf{e}}_1(n) + \eta\hat{\mathbf{e}}_2(n)$. Here $\hat{\mathbf{e}}_1(n)$ and $\hat{\mathbf{e}}_2(n)$ are the unit vectors of J_n in the expanding and contracting directions, respectively. The scaling exponents are now introduced by the *Ansatz*

$$P_n(l_1,l_2) \sim l_1^{\alpha_1} l_2^{\alpha_2}, \tag{3}$$

where the relation to the more common isotropic scaling $P(l) \sim l^\alpha$ is obtained when $l_1 = l_2$, leading to $\alpha = \alpha_1 + \alpha_2$. Sets that have a spectrum of scaling exponents α are referred to as multifractals; a convenient characterization of a multifractal is via the function $f(\alpha)$ which measures how many times, $N(\alpha)\Delta\alpha$, one finds the scaling exponent falling in an interval of size $\Delta\alpha$, $N(\alpha)\Delta\alpha \sim l^{-f(\alpha)}\Delta\alpha$. One of our goals is to calculate systematically the function $f(\alpha)$ for typical attractors. Numerical attempts to do so have been quite abundant recently[10]; we shall make a point here that the systematic approach presented below sheds light on the deficiencies of the direct numerical computations and culminates with a satisfactory understanding of an $f(\alpha)$ curve for typical attractors. This $f(\alpha)$ turns out to be rather nontrivial, reflecting contributions from orbits that are close to marginality, which in turn are tied to the nonhyperbolic nature of typical attractors.

The idea is now to use the periodic orbits to solve for the scaling exponents α_1 and α_2. One relation between the Lyapunov numbers of the orbits and their scaling exponents is easily obtained.[5] We make use of the fact that close to any point $\mathbf{x}_n$ of the attractor there exists a point belonging to some periodic orbit of length, say, m. Thus $\mathbf{x}_{m+n} = \mathbf{x}_n$, and from (2)

$$P_{n+m}(l_1,l_2) \approx P_n(l_1,l_2). \tag{4}$$

On the other hand, after m iterations the original box $l_1 \times l_2$ has been deformed to one of size $l_1 \exp(\lambda_1^{(m)}) \times l_2 \exp(\lambda_2^{(m)})$. With use of the preservation of probability

$$\begin{aligned} P_{n+m}(l_1,l_2) &= P_n(l_1 e^{-\lambda_1^{(m)}}, l_2 e^{-\lambda_2^{(m)}}) \\ &= l_1^{\alpha_1} l_2^{\alpha_2} e^{(-\alpha_1\lambda_1^{(m)} - \alpha_2\lambda_2^{(m)})}, \end{aligned} \tag{5}$$

where in the last step Eq. (3) has been used. Collecting Eqs. (3)-(5) we conclude that

$$\lambda_1^{(m)}\alpha_1 + \lambda_2^{(m)}\alpha_2 = 0 \tag{6}$$

for any cycle of length m.

For hyperbolic attractors, when the measure is uniform in the expanding direction, Eq. (6) is sufficient for the calculation of the $f(\alpha)$ spectrum.[5b] There $\alpha_1 = 1$ and hence we have an equation for α_2. We can therefore calculate all the α values obtained from all the orbits of length m for increasing m, collect the α values into bins, and count how many times one finds values in each bin, from which $f(\alpha)$ can be obtained. For typical attractors it is incorrect that $\alpha_1 = 1$; the measure can exhibit complicated singularities in the expanding direction, and a second equation for α_1 and α_2 is needed.

To proceed we have to equip the orbits with a symbolic address. Usually this is done by our finding a partition of space into k parts ($k > 1$) such that each orbit can be uniquely associated with one succession of symbols (an itinerary) representing the parts that it visits in time. Since the symbolic representation depends generally on the dynamical system in question, we focus on the example of the Hénon map $(x,y) \to (y + a - x^2, bx)$, with $a = 1.4$ and $b = 0.3$.[11] It is believed that the complex orbit seen numerically at these parameters has all the typical difficulties which are expected to be found in the phenomenology of strange attractors. For example, a partition of the plane by the y axis is known not to yield "good" symbolic dynamics; such a simple partition results in different periodic orbits having the same itinerary.

A way to overcome this difficulty is to choose a partition which passes through the homoclinic tangencies.[3,12] This is done by our refining the partition to pass between the points of two cycles with the same itinerary which sit closest to the present partition. The way that this is done in practice is the following: Beginning with the y axis as a partition we find the symbol sequences of all the orbits until at some period k we discover two orbits with the same itinerary. An examination of the points of these orbits closest to the y axis results in the discovery of a homoclinic tangency between them. The partition is then refined to pass between these points, and this partition is used until the next degeneracy occurs. Then the process is repeated. Having a partition, we define $\chi(\mathbf{x}) = 0$ if $\mathbf{x}$ lies on the left of the partition, and $\chi(\mathbf{x}) = 1$, if it lies on the right. We have ample numerical evidence that all the periodic orbits receive a unique itinerary in this scheme.

We are now in a position to derive the central result of this Letter, which is the second relation between the Lyapunov numbers and the scaling exponents. Assume that we have found all the periodic points of length n. We use the stable eigenvalues, $\exp(\lambda_2^{(m)})$, to define the local scale of an array of strips that cover the attractor.[5] We want, however, to find another array of strips to achieve a grid that covers the attractor [see Fig. 1(a)]. We do so by noticing that the same periodic points are shared by the map run backwards. We can then use the stable eigenvalues of the backwards map, i.e., $1/\exp(\lambda_1^{(m)})$ to define scales of strips "orthogonal" to the first array.[3] The horizontal strips can also be identified by the itinerary of the periodic point in them.[3] The intersection of the horizontal and vertical strips define boxes which can be labeled by the pair $(\mathbf{H},\mathbf{T})$, where $\mathbf{H}$

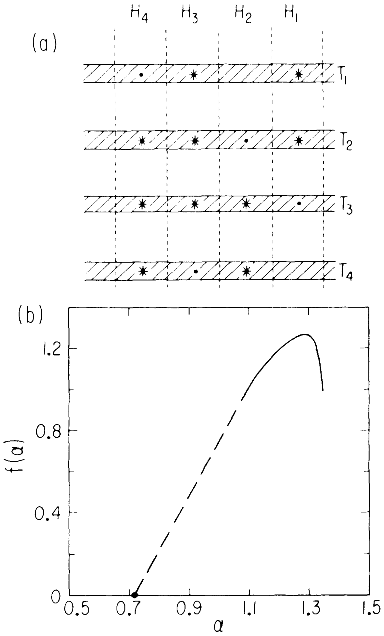

FIG. 1. (a) Schematic representation of the partition of the attractor by horizontal and vertical strips. Horizontal strips are denoted $\mathbf{T}_i$ which are the itineraries of the periodic points belonging to orbits of length m which are used to determine the width of the strips. The vertical strips are denoted by $\mathbf{H}_i$ which are also the itineraries of the periodic points in them. Notice that the order of horizontal strips is different from that for the vertical ones. Here $\mathbf{H}_4=\mathbf{T}_1$, $\mathbf{H}_2=\mathbf{T}_2$, $\mathbf{H}_1=\mathbf{T}_3$, and $\mathbf{H}_3=\mathbf{T}_4$. The points of the orbits of length m are plotted as dots, whereas those of length $2m$ are plotted as stars. (b) The $f(\alpha)$ curve for the Hénon attractor determined from the eigenvalues of the orbits of periods 10 and 20. The end of the smooth part of the curve and the isolated point on the left have been joined by a straight line.

and $\mathbf{T}$ stand for the itineraries of the periodic points in the horizontal and vertical strips, respectively.[3]

What remains now is to estimate the measure of each box. Evidently not every box contains at this stage periodic points; in fact, most of them are empty. To overcome this we look now at all the periodic points of length $2m$. Each of these is equipped with an itinerary of $2m$ symbols, which we consider as a head and a tail of m symbols, respectively. By matching the head and tail of the address to vertical and horizontal strips, respectively, we assign a periodic point of length $2m$ to any box that matches the itinerary of the $2m$ points. We know that there can be at most one point in each box, and some boxes are still left empty. The empty boxes are assigned zero measure, whereas the boxes that contain a periodic point are assigned a probability proportional to $\exp(-\lambda_1^{(2m)})$. With use of Eq. (3)

$$(e^{-\lambda_1^{(m)}(H)})^{\alpha_1}(e^{\lambda_2^{(m)}(T)})^{\alpha_2}\sim e^{-\lambda_1^{(2m)}(H,T)}, \tag{7}$$

where H and T stand for the head and the tail of the itinerary. This equation can be written as

$$-\lambda_1^{(m)}(H)\alpha_1+\lambda_2^{(m)}(T)\alpha_2\approx -\lambda_1^{(2m)}(H,T). \tag{8}$$

At this point we rewrite Eq. (6) as

$$\lambda_1^{(2m)}(H,T)\alpha_1+\lambda_2^{(2m)}(H,T)\alpha_2=0, \tag{9}$$

since the boxes are defined about $2m$-orbit points.

The difference between hyperbolic and nonhyperbolic attractors can be very sharply seen from these equations. In hyperbolic systems, where none of the orbits are close to marginality, we have roughly $\lambda_1^{(2m)}\sim 2\lambda_1^{(m)}$. Substituting this in Eq. (9) and subtracting from Eq. (8) results in $-2\lambda_1^{(m)}\alpha_1\approx -\lambda_1^{(2m)}\approx -2\lambda_1^{(m)}$ or $\alpha_1=1$. Thus, the only way that we can see interesting singularities in the expanding direction is that there appear values of $\lambda_1^{(2m)}$ which are significantly smaller than $2\lambda_1^{(m)}$. This occurs if the orbit is becoming marginally stable, i.e., much less unstable than typical orbits.

This phenomenon certainly occurs in the Hénon map. We have found all the periodic points up to length 20, using methods that were described before.[5] Next we solved Eqs. (7) and (8) using orbits of lengths 6-12, 7-14, 8-16, 9-18, and 10-20. Nothing curious happens when the 6-12 and 7-14 data sets are used. We find a continuous range of $\alpha=\alpha_1+\alpha_2$, where the maximum value of α is always determined by the fixed point, which among all the orbits has the largest ratio of $|\lambda_1^{(m)}/\lambda_2^{(m)}|$. Also α_1 is about 1 all the time.

When we go to the 8-16 data sets, anomalies appear. There are two anomalous period-16 orbits with atypically small $\lambda_1^{(16)}$. These orbits contribute anomalously small α_1 values, $\alpha_1\approx 0.59$ with a total α value ≈ 0.72. The interesting point is that there is now a *definite gap* in the α spectrum. Almost all the α values are greater than 1, but there are isolated values of α contributed by the anomalous orbits. When we go to the 9-18 data set the phenomenon disappears, returning in the 10-20 data set, again as isolated points. Evidently, in the $f(\alpha)$ language, these isolated α values correspond to $f=0$. It is thus interesting to examine the total $f(\alpha)$ function.

To calculate the $f(\alpha)$ function efficiently, we turn to the generalized dimensions D_q and their counterparts $\tau(q)=(q-1)D_q$ from which the $f(\alpha)$ can be found as a Legendre transform.[8] To form the partition sum $\Gamma(q,\tau)=\sum p_i^q/l_i^\tau$, from which $\tau(q)$ is found by solving $\Gamma(q,\tau)=1$, we focus again on Fig. 1(a). We have there an array of boxes of sizes $\exp[-\lambda_1^{(m)}(H)]\times\exp[\lambda_2^{(m)}(T)]$, each of which is equipped with a measure $\exp[-\lambda_1^{(2m)}(H,T)]$. We can cover each of these boxes by $\exp[-\lambda_1^{(m)}(H)]/\exp[\lambda_2^{(m)}(T)]$ square boxes of edge $\exp[\lambda_2^{(m)}(T)]$, each with a measure $\exp[-\lambda_1^{(2m)}(H,T)]$ divided by the number of these squares boxes. These are the p_i and l_i that are used in the partition sum, from which $\tau(q)$ is found. $f(\alpha)$ then satisfies $f(\alpha)=q\tau'(q)-\tau(q)$.[8]

In Fig. 1(b) the continuous curve represents the $f(\alpha)$

function obtained from the data set of 10-20 periods, and we believe that it is well converged. $\alpha_{\max}$ is the value expected from the contribution of the fixed point, and $f(\alpha_{\max})=1$ (this is the dimension of the piece of the manifold on which the fixed point is located). The maximum f is close to the Hausdorff dimension $D_0=1.27\ldots$ as is should be, and the curve is tangent to the $f(\alpha)=\alpha$ diagonal at $q=1$ as is expected on general grounds.[8]

On the other hand, the $f(\alpha)$ function fails to go below $\alpha=1$, which is a value actually reached already between $q=2.2$ and $q=2.4$. The reason is that in the partition sum the isolated measures contributed by the anomalous cycles are distributed over a considerable number of little square boxes as described above and this singular contribution is smeared out. We possess, however, from the solution of Eqs. (7) and (8) the α values that they contribute, and we plot them on the $f=0$ axis of Fig. 1(b). It appears that we can connect the isolated point with a straight line to where the $f(\alpha)$ curve is accumulating, and interpret the phenomenon as a phase transition.[13] This interpretation is not in contradiction with a conjecture by Grassberger, Badii, and Politi[14] that there is indeed a phase transition at $q\approx 2.3$. We point out, however, that there is no guarantee that higher-order cycles would not contribute anomalous, isolated α values which are smaller or larger than the ones that we found to the order that we have reached. Moreover, we feel that this $f=0$ component is extremely sensitive to the parameters a and b of the map since anomalous cycles can be shed off the attractor, or others may become anomalous, upon minute changes in the parameters.[15] If there exists a phase transition, it is closer to a spin-glass transition with a variety of "ground states" (all the marginal cycles) rather than to a single well-defined ground state. It should be clear now that direct box-counting or pair-counting algorithms are not very likely to discover a phenomenon of this type. Rather, a smooth curve is likely to be found even for values of α smaller than 1,[10] and the interesting physics occurring here can be completely missed. This is but another example that the calculation of an $f(\alpha)$ function should not be a goal in itself but rather be a result of the understanding of the multifractal measure.

To conclude, we argue that a good understanding of the scaling properties of the invariant measure of chaotic attractors can be obtained by looking at the periodic orbits. Rather than considering the attractor as a nonlinear continuation of the unstable manifold of the fixed point, we construct it from many linear bits of manifolds of periodic points, with local scales estimated from their eigenvalues. To stress the fact that Eq. (1) is a reasonable approximant to the invariant measure, we show that it can be used to calculate ergodic averages. As an example we calculate the Lyapunov exponent of the map. According to Eq. (1), this can be estimated to be

$$\langle\lambda_1\rangle=\frac{\sum_{\mathbf{y}\in\mathrm{fix}m}\lambda_1^{(m)}(\mathbf{y})\exp[-\lambda_1^{(m)}(\mathbf{y})]}{\sum_{\mathbf{y}\in\mathrm{fix}m}\exp[-\lambda_1^{(m)}(\mathbf{y})]}.$$

For $m=18$, 20, and 22 we obtain a Lyapunov exponent of 0.413, 0.404, and 0.433, which is very close to the estimated result of 0.419.[16] We feel that we can safely conclude that organizing chaotic motion around periodic orbits seems a reasonable thing to do.

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