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Ergodic Adiabatic Invariants of Chaotic Systems

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For a slowly time-dependent Hamiltonian system exhibiting motion which ergodically covers the energy surface, the phase-space volume enclosed inside this surface is an adiabatic invariant. In this paper, the scaling of the error in the adiabatic approximation is investigated for this situation via numerical experiments on chaotic billiard systems. It is found that the scaling depends on the long-time behavior of correlations in the chaotic system.

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We consider a conservative dynamical system characterized by a time-dependent Hamiltonian $H(\mathbf{p},\mathbf{q},\epsilon t)$, where $\mathbf{p}$ and $\mathbf{q}$ are N-vectors, and the explicit time dependence of H is "slow." To emphasize this slowness, we have written the third argument of H as ϵt, where we shall formally take ϵ small. Alternatively, we can set $T=\epsilon^{-1}$ and think of T as the time scale over which $H(\mathbf{p},\mathbf{q},\epsilon t)$ goes through an order-1 change, $T^{-1} \simeq H^{-1}\partial H/\partial t$. The statement that this time dependence is slow (or adiabatic) is equivalent to saying that T is much longer than any relevant characteristic time for the particle motion in the "frozen" Hamiltonian, $H(\mathbf{p},\mathbf{q},\epsilon t_0)$, where t_0 is a *constant*. Here we shall consider an adiabatic invariant for such a system. We presume that the number of degrees of freedom is greater than one, and that motion in the frozen Hamiltonian is chaotic and ergodic on the constant-energy surface, $H(\mathbf{p}(t),\mathbf{q}(t),\epsilon t_0)=\text{const}$. Consequently, the motion has no additional isolating constant of the motion other than the frozen Hamiltonian itself. In this case, as shown subsequently, the volume enclosed within the surface of constant H is an adiabatic invariant. (This presupposes, of course, that this volume is finite.) This case of an adiabatic invariant for $N \gg 1$ has been known for a long time within the context of statistical mechanics.[1] The volume inside the constant H surface is

$$\mu(E,t)=\int\int U[E-H(\mathbf{p},\mathbf{q},\epsilon t)]d^N p\, d^N q, \qquad (1)$$

where $U[\dots]$ denotes the unit step function, and E is the energy. Thus, for example, given an initial condition and the corresponding energy $E=E_0$ at $t=0$, calculation of $\mu(E,t)$ from (1) allows us to obtain an approximation to the energy $E(t)$ at all subsequent times via $\mu(E,t)=\mu(E_0,0)$. We call $\mu(E,t)$ for $N>1$ the *ergodic adiabatic invariant*.[2] To see how the approximate invariance of the quantity given by (1) follows from Hamilton's equations, we note that if *any* closed surface is specified at $t=0$ and each point on that surface is evolved in time with use of Hamilton's equations, then the new surface must enclose the same $2N$-dimensional phase-space volume as the initial surface.[3] If a particle wanders ergodically over the $H(\mathbf{p},\mathbf{q},\epsilon t_0)=E$ surface in a time that is short compared with T, then, as t increases, particles on an initial $H=\text{const}$ surface will all have qualitatively similar trajectories. In particular, their subsequent energies will be approximately equal. Thus, an initial $H=\text{const}$ surface [Fig. 1(a)] evolves into another surface [the squiggly surface in Fig. 1(b)] which is close to an $H=\text{const}$ surface [the smooth surface in Fig. 1(b)]. Hence, (1) is an adiabatic invariant, i.e., the squiggly surface in Fig. 1(b), in some sense, approaches the smooth $H=\text{const}$ surface as $\epsilon\rightarrow 0$. Our main concern in this paper is to obtain an estimate of how the error in the adiabatic invariant, as measured by the average distance between these two surfaces, scales with ϵ. It is found that this scaling depends on the long-time behavior of correlations for the ergodic particle motion.

With the advent of computer solutions for particle motion, it has become more and more appreciated that few-degree-of-freedom Hamiltonian systems can often

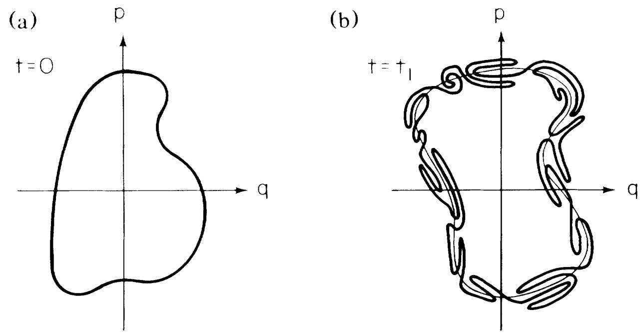

FIG. 1. (a) Initial $H=E_0$ surface at $t=0$ evolves under the exact dynamics into the convoluted surface shown in (b) which is close to an energy surface $H=E(t_1)$, where $E(t_1)$ is obtained from the constancy of μ.

behave chaotically in such a way that particle motion samples the surface of constant H, if not fully, at least nearly fully. Thus the ergodic adiabatic invariant is of interest not only for the $N\rightarrow\infty$ limit of statistical mechanics, but also for low N. This seems to have first been appreciated within the context of plasma physics where it was used by Wong *et al.*[4] in formulating a proposed magnetic plasma-confinement concept, and by Lovelace[5] to analyze the compression of a plasma ring confined by large-orbit gyrating ions.

At this point it may be instructive to discuss an example of the ergodic adiabatic invariant. Consider the situation shown in Fig. 2, where a point particle P of mass m moves in a two-dimensional square container with impenetrable walls of dimension L in the center of which is situated an impenetrable circular barrier of radius r. The dynamics is specified by the constancy of the particle velocity between encounters with the boundary at which the particle reflection is elastic. The motion of P in this "billiard" is known[6] to be chaotic and ergodic on the energy surface. For a billiard ($N=2$), the ergodic adiabatic invariant given by Eq. (1) becomes

$$\mu=2\pi mEA, \tag{2}$$

where A is the area of the billiard. Thus, for example, if one of the dimensions, L or r, is varied slowly with time, then the variation of E would be determined by the constancy of μ, $E(t)=E(0)A(0)/A(t)$. This result also has an intimate connection to the adiabatic gas law, $pV^\gamma=$const, where V is the volume, $\gamma=(\tilde{N}+2)/\tilde{N}$, $\tilde{N}$ is the number of degrees of freedom, and p is the pressure $p=nk\tilde{T}$ [with n the particle density and $(\tilde{N}/2)k\tilde{T}$ the average energy of a gas particle]. Now consider the particle in Fig. 2, and treat it as if it were a gas. Since $N=\tilde{N}=2$, we have $\gamma=2$ and $V=A$. Also, since we only have one particle, $n=1/A$ and $k\tilde{T}=E$. Thus $pV^\gamma=EA$, so that constancy of pV^γ implies constancy of EA and, hence, μ. Thus the single chaotic particle behaves like an adiabatic gas.

An essential question is *how good is the ergodic adiabatic invariant*; or, more specifically, what is the error incurred in the statement $\mu=$const? In the case of $N=1$ and periodic motion,[2] there is an adiabatic invariant (whose asymptotic series has μ as its first term) for which the error is smaller than any power of ϵ for sufficiently smooth time variation of H. The case of the ergodic invariant with $N\geq 2$ has been considered theoretically by Ott[7] using a multiple–time-scale expansion. The main result is an estimate of the typical rms error incurred by the approximation. The error-estimate result in Ref. 7 depends on two *hypotheses*: (a) The particle orbit in the frozen Hamiltonian is ergodic on the energy surface, and (b) a certain correlation function $C(t)$ is integrable, $\int_0^\infty C(t)\,dt<\infty$. Thus, for example, from (b), the derivation of Ref. 7 does not apply if the relevant correlation function has a long-time tail $C(t)\sim t^{-1}$ for $t\rightarrow\infty$. An analysis for the case $C(t)\sim t^{-1}$ will appear elsewhere.[8] The setting of Ref. 7 is that of Hamiltonians with smooth dependence on $\mathbf{p}$ and $\mathbf{q}$. However, it can be shown (Ref. 8) that similar results apply for the case of chaotic billiard problems. Combining the theoretical results of Ref. 7 with those in Ref. 8, we have the following: (i) If hypotheses (a) and (b) are satisfied, then the typical rms error in the ergodic adiabatic scales as $\epsilon^{1/2}$ for small ϵ, and this applies both for smooth $(\mathbf{p},\mathbf{q})$ variation (Ref. 7) and for billiards (Ref. 8). (ii) If hypothesis (a) is satisfied, but (b) is violated with $C(t)\sim 1/t$ for large t, then the typical rms error scales as $(\epsilon\ln\epsilon^{-1})^{1/2}$ (Ref. 8).

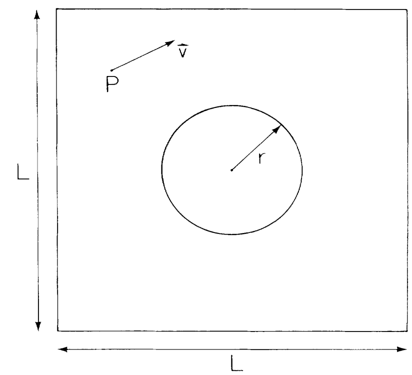

FIG. 2. Square billiard.

Note, for example, that $C(t)\sim 1/t$ for the billiard example in Fig. 2. The existence of this type of long-time tail for a correlation function in the situation of Fig. 2 has been shown by Zacherl *et al.*[9,10] A modification of the billiard in Fig. 2, for which hypothesis (b) is satisfied, is shown in Fig. 3.

By contrasting the theoretical results mentioned above [namely, error scalings like $\epsilon^{1/2}$ and $(\epsilon\ln\epsilon^{-1})^{1/2}$] with the results for the case of periodic motion[2] in $N=1$ [where the error can be smaller than $O(\epsilon^m)$ for all finite m], we see that the adiabatic invariant approximation is, in general, not as good for chaotic motion ($N\geq 2$) as compared with the case of periodic motion with $N=1$.

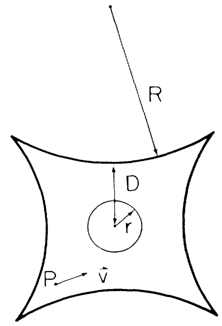

FIG. 3. Billiard with convex walls.

It is of interest to perform numerical experiments testing these results.[11] [Indeed we were only led to result (ii) by the outcome of our numerical experiments on the billiard of Fig. 2.] The numerical experiments to evaluate the average error in the adiabatic invariant are done as follows. At $t=0$, a large number M of particles are given positions and momenta assigned randomly with a uniform distribution on the energy surface. In the case of billiards, Figs. 2 and 3, this corresponds to uniform distribution in the accessible area and uniform distribution in angular direction of the velocities; the velocity magnitudes are all equal (corresponding to a constant energy). The size and shape of the billiard are then slowly varied and the particle orbits calculated. At some final time, the particle energies E_i are calculated (here i is a particle label). For each E_i, a corresponding μ_i is obtained, $\mu_i = 2\pi m E_i A$ [cf. Eq. (2)], and compared with μ_0, the initial value of μ. The root mean square error is then calculated from

$$\sigma = \left[\frac{1}{M} \sum_{i=1}^{M} (\mu_i - \mu_0)^2 \right]^{1/2}. \tag{3}$$

Numerical experiment on the billiard of Fig. 3.—The slow time dependence is added by our deforming the billiard walls by varying the distance D (Fig. 3) in time t according to $D = D_0 - \Delta D \cos(2\pi t/T)$, while keeping the radii r and R fixed in time. For our numerical experiments we choose $r=1$, $D_0=2$, $\Delta D=0.5$, $R=6.297$, and an initial particle speed of $|\mathbf{v}|=1$. At reflections from a moving wall the particle's velocity is changed from $\mathbf{v}_-$, before hitting the wall, to $\mathbf{v}_+$ after hitting the wall via $\mathbf{v}_+ = \mathbf{v}_- + 2[(\mathbf{w}-\mathbf{v}_-)\cdot\mathbf{n}]\mathbf{n}$, where $\mathbf{n}$ is the inward unit normal to the wall at the point of impact, and $\mathbf{w}$ is the local wall velocity. We calculate σ at $t=T/2$, as well as at $t=T$, for a range of T values and investigate its dependence on the slowness T. (Note that since $|\mathbf{v}| \simeq 1$ and $D \simeq 1$, T is of the order of the number of wall reflections experienced by the particle during the billiard oscillation.) The results are shown in Figs. 4(a) and 4(b). The straight lines are least-square-error lines of best fit for the $T \geq 1000$ data. The data indicated by crosses and circles correspond to σ^2 evaluated at $t=T$ and $t=T/2$, respectively. An ensemble of $M=50000$ particles was used. The abscissa is the slowness parameter $\epsilon \equiv T^{-1}$, while the ordinate is $\hat{\sigma}^2 \equiv \sigma^2/\mu_0^2$. Figure 4(b) is an enlargement of the region in Fig. 4(a) near the origin showing the data for $T \geq 1000$. It is evident that the theoretically expected asymptotic linear relationship between $\hat{\sigma}^2$ and ϵ is found to hold throughout almost the entire range of ϵ.

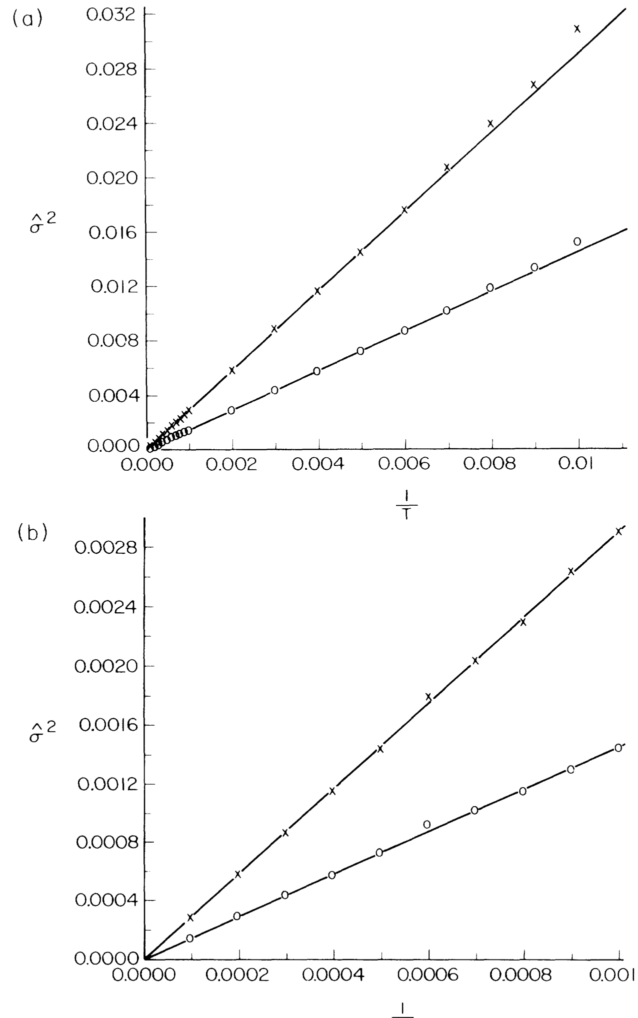

FIG. 4. Results for the numerical experiment on the billiard of Fig. 3.

Numerical experiments on the billiard of Fig. 2.—In this case the slow deformation is accomplished by variation of L, with r held fixed in time, $L/2 = L_0 - \Delta L \cos(2\pi t/T)$, where we choose $L_0=2$, $\Delta L=0.5$, $r=1$, $M=5000$, and an initial speed $|\mathbf{v}|=1$. The existence of a long-time $1/t$ tail in the correlation function for the (time-independent) billiard in Fig. 2 has been demonstrated by Zacherl *et al.*[9] Roughly this long-time tail arises because the periodic orbit represented by a particle moving exactly in the vertical (or equivalently horizontal) direction (and not hitting the circle) has neu-

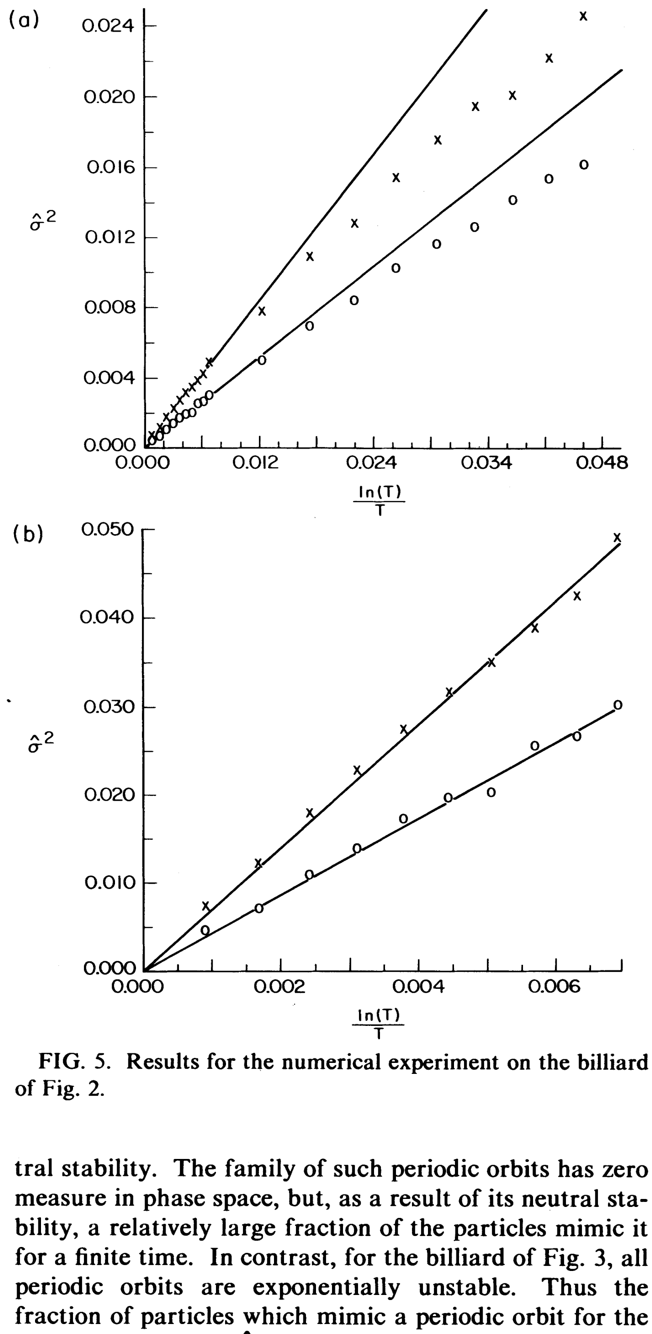

FIG. 5. Results for the numerical experiment on the billiard of Fig. 2.

tral stability. The family of such periodic orbits has zero measure in phase space, but, as a result of its neutral stability, a relatively large fraction of the particles mimic it for a finite time. In contrast, for the billiard of Fig. 3, all periodic orbits are exponentially unstable. Thus the fraction of particles which mimic a periodic orbit for the Fig. 3 billiard for $\hat{t}$ or more bounces is exponentially small in $\hat{t}$, and a long-time tail is absent. The results of our numerical experiments on the billiard of Fig. 2 are shown in Figs. 5(a) and 5(b). In these figures the abscissa is $\epsilon \ln(\epsilon^{-1}) \equiv \ln(T)/T$, while the ordinate is again $\hat{\sigma}^2 \equiv \sigma^2/\mu_0^2$. Again the crosses denote data taken at $t=T$, while the circles denote data taken at $t=T/2$, and the straight lines are least-square-error lines of best fit for the data in the asymptotic region ($T \geq 1000$). We expect that for large T the quantity σ^2 will depend linearly on $\ln(T)/T$. Note that, in comparison with the example of Fig. 3, much larger T values are required before σ^2 is accurately fitted by its asymptotically predicted behavior. That is, for Fig. 4, the straight-line fit is good for $T \geq 200$, while for Fig. 5, $T \geq 1000$ is required. This is not unexpected since, as a result of the predicted forms of the error in the two cases, we might guess that the conditions for the asymptotic regimes to apply are $T \gg 1$ for Fig. 3 and $\ln(T) \gg 1$ for Fig. 2.

In conclusion, we have examined the scaling of the error in the ergodic adiabatic invariant of chaotic particle motion using numerical experiments on billiards. When correlations decrease sufficiently fast with time, the size of the average error scales as $\epsilon^{1/2}$. When the correlation function has a long-time $1/t$ tail, the error behavior is less favorable and scales as $(\epsilon \ln \epsilon^{-1})^{1/2}$.

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[1]For example, see the text by R. Kubo, *Statistical Mechanics* (North-Holland, Amsterdam, 1965), p. 14.

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