Static Potential for Smooth Strings

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The static potential is studied in Helfrich and Polyakov's model of smooth strings. For large separations R, the potential is linear in R with corrections of order $1/R$. While the physical string tension is renormalized by the extrinsic curvature coupling, the coefficient of the $1/R$ term has Lüscher's universal value to any finite order in the loop expansion. For very small separations, the leading term in the potential is proportional to $1/R$, with a coefficient exactly twice that of the Lüscher term, and with corrections that are logarithmically small.

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There are many indications that color flux tubes in QCD can be described by an effective theory of strings. Polyakov[1] has suggested that, for a realistic string model of hadrons, the Nambu action must be supplemented by a term that depends on the extrinsic curvature of the world sheet. This term favors smooth string configurations over those that are sharply curved, and was first investigated in the context of fluid membranes by Helfrich and others.[2] There have been several investigations of its effects in string theories.[3] In this note we study the static potential which would be generated between a heavy quark and antiquark if the color flux tube were modeled by a smooth string.

The action for this model of smooth strings is

$$S=M_0^2\int(\det g)^{1/2}d^2\xi+(1/2e^2)\int(\det g)^{1/2}[\Delta(g)X^\mu]^2d^2\xi. \tag{1}$$

The string coordinate X^μ is a vector in D (Euclidean) dimensions, $g_{ab}=\partial_a X^\mu \partial_b X_\mu$ is the induced metric on the surface, and $\Delta(g)$ the corresponding Laplacian. The first term in S is the Nambu action, with bare string tension M_0^2. The second is the extrinsic curvature term introduced by Helfrich and Polyakov. The coupling constant e^2 for the curvature term is dimensionless and asymptotically free.[1,2]

The behavior of the static potential $V(R)$ for the Nambu model is an old story. Classically, it is linear: $V=M_0^2R$. The parameter of the loop expansion is $1/M_0^2R^2$, so that the one-loop correction is proportional to $1/R$. It was first computed by Lüscher, Symanzik, and Weisz,[4] and on very general grounds is expected to be universal.[5,6] The loop expansion can be used to calculate $V(R)$ at large R, but for small R a nonperturbative method is required. Alvarez[7] showed that to leading order in $1/D$, where D is the number of dimensions, $V(R)$ is imaginary for R less than a critical distance R_c. This behavior was confirmed by Arvis,[8] who obtained the exact solution for $V(R)$ in the critical dimensionality, $D=26$.

To understand the nature of the static potential for smooth strings, we start by expanding to one-loop order about a flat surface. Amazingly, this gives us the leading behavior of the potential at both very large and very small distances. Choose a physical gauge[7] in which the world sheet is parametrized by the distances in space (r) and time (t). Consider fluctuations of the world sheet about a flat surface of length T and width R, for $T\gg R$. In such a physical gauge, ghosts can be ignored. Integration over the $D-2$ transverse degrees of freedom gives

$$V(R)=M_0^2R+[(D-2)/2T]\operatorname{tr}\ln[-e^{-2}\partial^2(-\partial^2+e^2M_0^2)], \tag{2}$$

where $-\partial^2$ is the Laplacian in flat space and the trace is over functions that vanish at $r=0$ and $r=R$. The divergent trace can be defined by analytic regularization, which introduces a renormalization mass scale μ and yields

$$V(R)=M_0^2R\left\{1+\frac{e^2(D-2)}{4\pi}\left[-\frac{1}{3}\frac{1}{\Lambda^2}+\tfrac{1}{2}\ln\left(\frac{e\mu^2R^2}{4\pi^2}\right)+\gamma+2S(\Lambda)\right]\right\}, \tag{3}$$

where $\Lambda=eM_0R/\pi$, γ is Euler's constant, and $S(\Lambda)$ is the convergent sum

$$S(\Lambda)=\frac{1}{\Lambda^2}\sum_{n=1}^{\infty}\left[(n^2+\Lambda^2)^{1/2}-n-\frac{\Lambda^2}{2n}\right]. \tag{4}$$

Expanding in powers of $1/R$, we find that for large separations the static potential behaves like

$$V(R)=M^2R+V_0-\frac{\pi(D-2)}{24}\frac{1}{R}+\cdots, \tag{5}$$

where M^2 is the physical string tension and $V_0=-(D-2)eM/8$. The subleading terms in (5) arise from the finite-R dependence of the free energy for $D-2$ massless fields and $D-2$ fields of mass eM. The massive fields contribute the zero-point energy V_0, while the massless fields produce the $1/R$ term with a coefficient that is identical to that found by Lüscher, Symanzik, and Weisz[4] in other string models.

In higher orders, both M^2 and V_0 will be further renormalized by the dimensionless curvature coupling e^2. On dimensional grounds there could also be further corrections to the coefficient of the $1/R$ term. In fact, to any finite order in e^2, there are not such corrections. This follows essentially from Lüscher's original argument for the universality of this coefficient.[5] Consider the propagator for small transverse fluctuations in X^μ about a flat surface, which can be written as the difference of propagators for a massless and a massive mode:

$$\frac{e^2}{-\partial^2(-\partial^2+e^2M_0^2)}=\frac{1}{M_0^2}\left[\frac{1}{-\partial^2}-\frac{1}{-\partial^2+e^2M_0^2}\right]. \tag{6}$$

The infrared behavior of smooth strings is described by an effective theory of interactions between these massless and massive modes. For the Nambu model, the crux of Lüscher's argument was that corrections generated in higher orders of the loop expansion would be less infrared singular than those at two loops. He showed that the two-loop correction did not have a term proportional to $1/R$, and therefore the higher orders could not either. The extension to the case of smooth strings is immediate. It is necessary to add the massive modes and the new interactions generated by the curvature term. But both only produce diagrams that are *less* infrared singular than those of the Nambu model. For the massive modes, this is simply because they are massive. For the new interactions, it is because they necessarily involve two more derivatives than the corresponding Nambu terms. Presumably, a more elegant argument could be given by conformal field theory.[6] We have checked explicitly that the two-loop correction to the Lüscher term vanishes to leading order in $1/D$.[9] The results of this calculation are given below in (15).

How does the static potential behave for small separations? If R is very small, it should be possible in (2) to neglect the term $e^2M_0^2$, so that the $1/R$ term should be *twice* that at large R. Indeed, the small-R behavior of the one-loop result (3) is $V(R)\sim-\pi(D-2)/12R$. It is important to ask whether there are logarithmic corrections, such as $\ln R/R$, which might change the dominant behavior at small R. To answer this, we can set $M_0=0$, and view the theory as if it were a massless gas at a temperature proportional to $1/R$. This analogy is not exact, for the boundary conditions at finite temperature are not the same as those here, but these differences do not affect our qualitative conclusions. $V(R)$ is like the free energy at the temperature $1/R$, with the $1/R$ term as the ideal-gas term. With use of this analogy, it is possible to show that the free energy of this nonideal gas for $R\ll 1/\mu$ is of the form

$$V(R)\sim -R^{-1}[\pi(D-2)/12+a_1e^2+a_1a_2e^4\ln(\mu R)+\cdots], \tag{7}$$

where the coefficients a_1 and a_2 are constants that depend only upon the dimensionality D. As was of concern, a term $\ln R/R$ does appear at order e^4. Fortunately, we can invoke the known asymptotic freedom of the theory[1,2] to conclude that these logarithms are innocuous. At a distance R, the running coupling $e^2(R)$ is

$$e^2(R)=\frac{e^2}{1-(De^2/4\pi)\ln(\mu R)}, \tag{8}$$

so that the constant a_2 in (7) must be $a_2=D/4\pi$. The renormalization group can be used to resum the series as

$$V(R)\sim-\frac{1}{R}\left[\frac{\pi(D-2)}{12}-\frac{a_1}{a_2\ln(\mu R)}+\ldots\right]. \tag{9}$$

Thus corrections to the ideal-gas term of (8) are logarithmically small. The coefficient a_1 has been computed to leading order in the $1/D$ expansion,[9] and is given below in (18).

In passing, we note that a successful fit to charmonium spectroscopy is attained by the model of Eichten *et al.*[10] with the potential $V(R)=M^2R-k/R$. If we take the $1/R$ term at small R from (9), we find $k=\pi(D-2)/12=0.524$ for $D=4$. It is amusing that from spectroscopy, Eichten *et al.* find $k=0.52$.

We now turn to a large-D expansion of the potential for smooth strings to verify some of the general conclusions made previously. The large-D expansion has also been applied by Alonso and Espriu and by David[3] to investigate other aspects of the smooth-string model. This expansion is formulated by our taking the parameters of the theory to scale with D as follows: $M_0^2\sim D$, $e^2\sim 1/D$, and $\mu\sim D^{1/4}$. The need for rescaling of the renormalization scale parameter μ with D will become clear later. Following Alvarez's treatment of the Nambu string,[7] the large-D limit of the static potential $V(R)$ can be expressed as the saddle point of an effective potential $W(R)$ that depends on four functions of r: the deviations $\sigma_0(r)$ and $\sigma_1(r)$ of the diagonal components g_{00} and g_{11} of the metric from unity, and a pair of

Lagrange multiplier fields $\alpha_0(r)$ and $\alpha_1(r)$. This effective potential is

$$W(R)=\int_0^R dr\left\{M_0^2A+\frac{1}{2e^2}\frac{1}{A}\left[\frac{d}{dr}E\right]^2-\tfrac{1}{2}M_0^2(\alpha_0\sigma_0+\alpha_1\sigma_1)\right\}+\frac{D}{4\pi}\int_{-\infty}^{\infty}d\omega\,\mathrm{tr}\ln\mathcal{O}, \quad (10)$$

where the operator $\mathcal{O}$ is

$$\mathcal{O}=M_0^2\left[-\frac{d}{dr}\alpha_1\frac{d}{dr}+\omega^2\alpha_0\right]+\frac{1}{e^2}\left[-\frac{d}{dr}E\frac{d}{dr}+\omega^2\frac{1}{E}\right]\frac{1}{A}\left[-\frac{d}{dr}E\frac{d}{dr}+\omega^2\frac{1}{E}\right], \quad (11)$$

and A and E are combinations of σ_0 and σ_1 defined by $1+\sigma_0=AE$ and $1+\sigma_1=A/E$.

The physical string tension M^2 can be determined by study of the effective potential at infinite R, where it takes the form $W(R)=M^2R$. Translational and rotational symmetries guarantee that the saddle point will have the form $\sigma_0(r)=\sigma_1(r)=\sigma$ and $\alpha_0(r)=\alpha_1(r)=\alpha$, where σ and α are constants. The effective potential reduces to

$$W(R)=M_0^2R\{1+\sigma-\alpha\sigma\}+\frac{DR}{8\pi^2}\int d^2\omega\ln\left[M_0^2\alpha\boldsymbol{\omega}^2+\frac{1}{e^2}\frac{(\boldsymbol{\omega}^2)^2}{1+\sigma}\right]. \quad (12)$$

If the divergent integral is defined by analytic regularization, the result is

$$W(R)=M_0^2R\{1+\sigma-\alpha\sigma-\lambda\alpha(1+\sigma)[2\ln\alpha+\ln(1+\sigma)+2\ln(eM_0^2/\mu^2)-2]\}, \quad (13)$$

where $\lambda\equiv De^2/16\pi$. The saddle-point equations for α and σ can be solved order by order in λ. Evaluating (13) at the saddle point, we find

$$M^2=M_0^2\{1+\lambda(2-2l)+\lambda^2(4l^2-2l)+\ldots\}, \quad (14)$$

where $l=\ln(eM_0^2/\mu^2)$.

We next consider the static potential at finite R. The functions $\alpha_i(r)$ and $\sigma_i(r)$ at the saddle point of the effective potential (10) need no longer be independent of r. In the Nambu model, Alvarez showed that they were constant except for discontinuities at the end points $r=0,R$.[7] One should not expect this to remain true in the smooth string model, because the extrinsic curvature term is more sensitive to these discontinuities. It would be extremely difficult to find this saddle point for arbitrary coupling, and therefore we restrict out attention to small De^2. The large-D expansion is then no more than a convenient way of organizing certain terms in the loop expansion. We have computed the potential to two-loop order in this limit, but since the calculations are rather involved,[9] we merely quote the results here. At large R,

$$V(R)=M^2R+V_0-\frac{\pi D}{24}\frac{1}{R}-\frac{1}{8}\left[\frac{\pi D}{12}\right]^2\frac{1}{m^2R^3}\left[1-\frac{3\pi}{10}\frac{1}{eMR}+\cdots\right], \quad (15)$$

where the physical string tension M^2 is given in (14) and the constant term is

$$V_0=-(D/8)eM_0\{1+\lambda[3\pi-7-4l]\}. \quad (16)$$

As predicted, there are no corrections of order De^2 to the $1/R$ term.[11] Note also that if we set $e=\infty$ in (15), we recover the first few terms in the expansion of the large-D limit of the Nambu potential[7]

$$V(R)=M^2R[1-(\pi D/12)/M^2R^2]^{1/2}. \quad (17)$$

At small R, the potential to two-loop order is

$$V(R)=-(\pi D/12)(1+4\lambda)/R+M_0^2\{1+2\lambda L+\lambda^2[4L^2+8-\tfrac{2}{3}\pi^2-\tfrac{4}{3}\zeta(3)]\}R, \quad (18)$$

where $L=\ln(e\mu^2R^2/4\pi^2)+2\gamma$ and $\zeta(3)\simeq 1.202$. The correction to the coefficient of the $1/R$ term is indeed of the form (7) with $a_1=D/4\pi$ to leading order in $1/D$.

We have determined the behavior of the static potential for smooth strings, for both large R and small R, and verified our results in the large-D limit by explicit two-loop calculations. For large R, the coefficient of the $1/R$ correction to the linear potential has the universal value predicted by Lüscher, with no corrections to any finite order in the loop expansion. For small R, $V(R)$ behaves like $1/R$ with a coefficient that is twice Lüscher's value. The corrections to this coefficient are logarithmically suppressed by the asymptotic freedom of the extrinsic curvature coupling. Over intermediate distance scales, $V(R)$ will have a complicated dependence on the extrinsic curvature coupling e^2, but we see no reason why it should not be well defined and physical over all distance scales, as long as e^2 is sufficiently small. This is in marked contrast to the Nambu model ($e^2 = \infty$), where $V(R)$ is imaginary for R less than R_c. Since the existence of R_c in $D = 26$ can be related to the tachyon in the mass spectrum,[12] this suggests that the spectrum for smooth strings may be free of tachyons for small e^2.

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