Hydrodynamic Boundary Conditions and Cooling in Superfluid ^{3}He

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(Received 17 November 1986)

In their superfluid state, ^{3}He and ^{4}He need different hydrodynamic boundary conditions. These have been derived and applied to various experimentally relevant situations, especially cooling. While one surface transport coefficient, the Kapitza resistance, is sufficient to account for any cooling setup in ^{4}He, three are needed in ^{3}He.

PACS numbers: 67.50.Fi, 67.40.−w

Which, and how many, boundary conditions does one need to find the solution appropriate for a superfluid exchanging energy and momentum with its surroundings? This seemingly simple question has not, in fact, received much attention in the past. Take the case of vanishing mass flow, $g=0$: In normal fluids, all one needs is the value of the entropy flow f, which fixes the thermal gradient at the boundary, $f=-(\kappa/T)T'(0)$, and yields uniqueness of the solution to the hydrodynamic equations. In superfluids, one may expect on physical grounds that f alone should again be enough. However, there are two flow fields here, v_s and v_n, and one needs an additional boundary condition, such as a prescription on how to divide the entropy flow, $f=sv_n-(\kappa/T)T'$, between v_n and T'. If one can neglect dissipative terms, as is plausible in the study of sound propagation in ^{4}He II, one may follow Khalatnikov[1] to give the whole load of thermal transport to $v_n=f/s$. On the other hand, there is much to be said for the other extreme, $v_n(0)=0$ at the boundary, where T' carries the whole load. In fact, with $v_s=v_n=0$ at the interface, in its rest frame, Brand and Cross[2] were able to account for the large damping that is observed in U-tube experiments[3] of superfluid ^{3}He.

We have studied this question within the hydrodynamic framework: With the help of conservation laws and irreversible thermodynamics, we have derived the general structure of the boundary conditions, parametrized by three surface Onsager coefficients. In ^{4}He II, they combine to yield only one, the Kapitza resistance, for all conceivable experimental situations. And the boundary conditions reduce to those of Khalatnikov, supplemented by $T'(0)=0$. In superfluid ^{3}He, all three Onsager coefficients remain independent and hence relevant to the interpretation of different experimental situations. Especially, no one single Kapitza resistance can account for all cooling data. For instance, a negative Kapitza resistance becomes possible in certain situations.

We start with a brief discussion of the number of boundary conditions and the peculiar situation of the collective modes in the two-fluid hydrodynamics: Since we are studying the spatial boundary conditions, it is appropriate to take the Fourier transform of the linearized hydrodynamic equations[1] in time. In addition, we shall assume a one-dimensional geometry and neglect the transverse component of v_n. Then we are left with four coupled, ordinary, second-order differential equations, given by the equations of motion for ρ, s, g, and v_s. With neglect of dissipative terms, these equations are of first order, for which four boundary conditions are needed. They may be given by g and f, on each side, and will determine the four amplitudes of first and second sounds in both directions. Taking dissipative terms into account introduces unexpected problems. We can rewrite the above four differential equations into six of first order, with g, T, T', v_n, v_n', and the pressure p as the variables. (Note that $g''=-i\omega\rho'$, while p'' does not appear in the equations.) Therefore, we need six boundary conditions for this general case and two additional modes, which seem to be lacking. A more careful evaluation of the characteristic polynomial, however, shows that it is fourth order in ω and sixth in q. So we have four ω modes yet six q modes: In addition to the sounds, $q_1=\pm\omega/c_1$ and $q_2=\pm\omega/c_2$, we must have two that remain finite for vanishing frequency. [In other words, the linearized hydrodynamic equations have introduced an autonomic length scale here. This is not the case in normal fluids, whose polynomial is third order in ω and fourth in q: Its three ω modes, $\omega_1=\pm c_1 q$, $i\omega_T=D_T q^2$, correspond trivially to four q modes: $q_1=\pm\omega/c_1$ and $q_T=\pm(i\omega/D_T)^{1/2}$]. The two additional q modes of the superfluid are $q_s=\pm i/\lambda(\omega)$, where λ, to first order in ω, is given by

$$\lambda^{-2}=\left(\eta_{\mathrm{eff}}+\frac{s^2T}{\kappa}\right)\left[1+\frac{1}{2}i\frac{\mathrm{Im}\{c_2^2\}}{\mathrm{Re}\{c_2^2\}}\right]\alpha^{-1}. \qquad (1)$$

Terms of order $\mathrm{Im}\{c_1^2\}/\mathrm{Re}\{c_1^2\}$ have been neglected. Note also that $\mathrm{Im}\{c_2^2\}\sim\omega$. $\alpha=\frac{4}{3}\eta-2\zeta_1\rho+\zeta_2+\zeta_3\rho^2$ denotes a repetitive combination of shear viscosities defined as usual,[1] while $\eta_{\mathrm{eff}}=\eta/12d^2$ accounts for lateral damping[2] in a geometry not strictly one dimensional. Except for g and f, all the variables T, T', v_n, v_n', and p participate in this pair of modes, which we may refer to as the superfluid q mode, or sq mode. Since they alone survive the limit $\omega\rightarrow 0$, they reduce to the stationary solution, calculated by Saslow and Putterman[4] for an open geometry (i.e., $\eta_{\rm eff}=0$):

$$T-T_\infty=T_s^- e^{x/\lambda}+T_s^+ e^{-x/\lambda},\qquad sv_n=\bar\kappa(T-T_\infty)+f, \tag{2}$$

where $\bar\kappa=\kappa/T\lambda$ and $g,f=$const.

With T_s^- and T_s^+, we have now the correct number of six amplitudes, to be determined by six boundary conditions. Next, we shall derive the structure of these boundary conditions. The energy flux normal to the interface, to quadratic order[5] and with vanishing shear flow, is given as $Q_L=T_Lf_L+(\alpha v_n'+\beta g')j_s/\rho$ for the superfluid on the left (L), $x<0$, and $Q_R=T_Rf_R$ for the solid on the right (R), $x>0$. Here, $j_s=\rho_s(v_s-v_n)$, α is defined above, and $\beta=\zeta_1-\zeta_3\rho$. An additional summand[5] in Q_R, proportional to the longitudinal component of the stress field, has been set equal to zero, because since it is a measure of the energy required for displacing a particle from the bulk solid to the surface,[6] it should vanish at the interface. Besides, keeping it only leads to a boundary condition for the solid. Another summand $\sim g$ was eliminated by going to the rest frame of the interface. (Only when one deals with melting or evaporation[7] is the mass current across the interface nonzero.) The mass and entropy flow, g_0 and f_0, in the rest frame are related to the respective quantities of the lab frame by $g_0=g-\rho u=0$, $f_0=f-su=-\sigma j_s-\kappa T'/T$, where u is the interface velocity. Now, the energy flux in the rest frame is continuous across the interface; hence

$$R_s\equiv -T\Delta f_0=f_0\Delta T+(\alpha v_n'+\beta g')j_s/\rho, \tag{3}$$

where $\Delta A\equiv A_L-A_R$ and $A\equiv(A_L+A_R)/2$ for T and f_0, and R_s/T denotes the surface entropy production. Therefore, we can set

$$f_0=a\Delta T+c(\alpha v_n'+\beta g')/\rho,\qquad j_s=c\Delta T+b(\alpha v_n'+\beta g')/\rho, \tag{4}$$

with the surface Onsager coefficients $a,b>0$, and $c^2\leqq ab$. In this context, it is important to realize that the force $\alpha v_n'+\beta g'$ stems from a combination of the fluxes of the equations of motion for g and v_s. Its time-reversal property must therefore be counted[8] as that of $\dot g$ and $\dot v_s$. It is therefore even rather than odd, resulting in a symmetric Onsager matrix. Barring melting or evaporation, Eqs. (4) and $g=\rho u$ are the most general boundary conditions compatible with conservation laws and irreversible thermodynamics.

The simplest, and very instructive, application of these boundary conditions is to calculate the amplitude T_s^- of the stationary solution ($g=0$, $f=$const) in a semi-infinite ($x<0$, hence $T_s^+=0$) and open ($\eta_{\rm eff}=0$) geometry. Inserting Eqs. (2) in Eqs. (4) to eliminate v_n' and $j_s=-\rho v_n$, we obtain two equations for ΔT and T_s^-, with f as the driving force. They are easily solved to yield

$$T_s^-/f=-(a+c\sigma)/N,\qquad \Delta T/f=(\bar\kappa+b\sigma^2+c\sigma)/N, \tag{5}$$

where $N=\bar\kappa a+(ab-c^2)\sigma^2$. Note first that f (or alternatively T_R, the wall temperature) is the only external parameter here, and so our intuitive physical argument that once f is fixed, the solution must be unique, is indeed correct. Second, both $\kappa T'/T$ and sv_n are proportional to f; cf. Eqs. (2) and (5). Hence Eqs. (4) indeed provide the prescription on how to divide the heat load. Especially, for $f=0$, we have $T'=0$ and $v_n=0$. Third, with $f>0$ going to the right, we have $T_s^-<0$ and $\Delta T>0$, if $c=0$. This is the "normal" case: Close to the wall, the far-away constant temperature T_∞ decays exponentially, becomes $T_L=T_\infty+T_s^-$ at $x=0$, then jumps down by $\Delta T=T_L-T_R$, achieving a total (bulk) temperature difference of $\Delta_BT\equiv T_\infty-T_R=\Delta T-T_s^-$. Generally $c\neq 0$; then two "abnormal" cases are also possible: (1) exponential increase of the *sq* mode, $T_s^->0$, $\Delta T>0$, or (2) negative Kapitza resistance, $T_s^-<0$, $\Delta T<0$. However, $\Delta_BT>0$ always holds, and this is easy to understand. When we derived Eq. (3), the thickness of the boundary was not specified, and we could have taken it to include the exponential decay of the *sq* mode, which is the only place in the bulk where entropy is being produced. Then the *total* entropy production can be considered as an *effective surface* one, $\bar R_s=f_0\Delta_BT+(\alpha v_n'+\beta g')j_s/\rho$, where v_n', g', and j_s are to be taken at $x\gg\lambda$. With $v_n'=g'=0$, we have $\bar R_s=f\Delta_BT$ and $f=A\Delta_BT$. Since $\bar R_s>0$, we always have $\Delta_BT>0$ if $f>0$. By use of Eqs. (5) the effective Kapitza conductance A is identified to be

$$A=N/(\bar\kappa+b\sigma^2+a+2c\sigma). \tag{6}$$

An interesting conclusion that will be useful below follows from the above observation: If we are only interested in the physics of the helium bulk, $|x|\gg\lambda$, then we obviously could have worked with the much simpler, effective boundary conditions:

$$f_0=A\Delta_BT,\quad v_n'=0. \tag{7}$$

Next, we turn our attention to nonstationary solutions and finite geometry. From this point on, we need to distinguish between ^{4}He and ^{3}He. In ^{4}He II, λ is exceedingly small, $\lambda=0.1\ \mu$m at 1.2 K, larger at lower temperatures and smaller at higher. (And if λ approaches the size of a microscopic correlation length, the whole theory of course becomes inadequate.) At any reasonable frequency and dimension of the geometry, λ is by far the smallest length scale, and the hydrodynamic behavior at $x\leqq\lambda$ is really not accessible to experiments. Therefore, we can confine ourselves to $x\gg\lambda$ and again consider the effective surface entropy production $\bar R_s$. However, for $x\gg\lambda$, $j_s=-f_0/\sigma+O(\omega)$ is, to lowest order in ω, not independent of f_0, and we have

$$\bar{R}_s = f_0[\Delta T - (\alpha v_n' + \beta g')/s] + o(\omega^2).$$

The three boundary conditions are therefore

$$f_0 = A[\Delta T - (\alpha v_n' + \beta g')/s] + o(\omega^2),$$

$g=\rho u$, and the vanishing of the sq mode. These are, to zeroth order in ω, the Khalatnikov boundary condition of Eqs. (7). Now, the first-order terms are proportional to $q_{1,2}$, and compared to the zeroth-order ones $\sim\lambda^{-1}$, are vastly smaller. Therefore, Eqs. (7) with $g=\rho u$ are valid also for finite frequency and geometry.

Turning our attention now to ^{3}He, we shall concentrate on the typical superfluid behavior rather than the liquid-crystal or antiferromagnetic ones, or the interplay between them. So we shall assume that all the preferred directions are clamped by appropriate boundary conditions and do not participate in the dynamics. Even then, there are differences from ^{4}He, the major one being the much larger λ. It is estimated to be[9] 0.2, 4, and 700 cm at 2, 1, and 0.5 mK, respectively. Therefore, the deviations of the hydrodynamic solution from its bulk behavior become relevant and experimentally accessible. For stationary setups, the existence of the sq mode and the predictions of Eqs. (5) and (6), including possibly a negative Kapitza resistance, can be directly verified. A second difference from ^{4}He is the much larger q_2, making terms of higher order in ω more important. We shall give two examples of oscillatory experiments.

The first example is given by a layer of helium, sandwiched between two parallel plates of separation D. Now, what is the temperature distribution in the liquid if both plates have a temperature $T_p = T_0 + \Delta T_p e^{-i\omega t}$? The answer is simple enough for ^{4}He, in which $D \gg \lambda$. Since T is essentially spatially constant, we have $(T_p - T)A = -f = +\frac{1}{2}\rho D(\partial\sigma/\partial T)\dot{T}$, requiring the solution $T = T_0 + \Delta T_p e^{-i\omega t}/(1-i\omega\tau)$, where $\tau = \frac{1}{2}A^{-1}\rho D\,\partial\sigma/\partial T$. For ^{3}He and $D\sim\lambda$, we have to look for a nonuniform solution in terms of the sq mode and first and second sound, and determine these six amplitudes by the three boundary condition, Eqs. (4), on each interface. Retaining only terms of first order in ω, we obtain $T = T_0 + \Delta T_p e^{-i\omega t}(1+i\omega\tau)$, where with $\xi = \frac{1}{2}D/\lambda$ and $E=(ab-c^2)^{-1}$:

$$\tau(x) = \frac{\partial\sigma}{\partial T}\,\frac{\left[\dfrac{\alpha}{\rho\sigma^2} + (a+c\sigma)\dfrac{E\rho D}{2\sigma^2}\right]\left[1-\cosh\left[\dfrac{x}{\lambda}+\xi\right]\operatorname{sech}\xi\right] + \left[b+\dfrac{c}{\sigma}+\lambda\left[\dfrac{\rho^2}{\alpha}-\dfrac{2c}{\sigma D}\right]\tanh\xi\right]\dfrac{\rho D E}{2}}{1+(\rho^2\lambda/ab)(1+c^2E)\tanh\xi}. \tag{8}$$

Note that since T is a function of x, so is τ. When we take the limit $x/\lambda, D/\lambda \to \infty$, Eq. (8) reduces to the ^{4}He expression, with A given by Eq. (6).

The second example is given by a semi-infinite geometry with an oscillating entropy current $\delta f \sim e^{-i\omega t}$ and an immovable wall, $g=0$. To the lowest order ($\sim\omega^0$), we have the sq mode as given by Eq. (5) in addition to second sound, and no first sound at all. To the next order ($\sim\omega^1$), however, all three modes contribute to g; only their sum remains zero. This leads to a first-sound amplitude as given by

$$\delta P/\delta f = i\omega(\alpha+\rho\beta)[c_1^2/(c_1^2+c_2^2) - (H+cE)/(H+\sigma/\bar{\kappa})]/c_1 s, \tag{9}$$

where $H=(1+c^2E)/b\sigma$. A similar effect arises from the frequency-independent thermal expansion, which was neglected in the calculation of Eq. (9). Since they are approximately equal in magnitude at $\omega=25$/sec in ^{3}He, the experiment should be performed at higher frequencies. (For ^{4}He, the same magnitude would be achieved for the unrealistic frequency of $\omega\sim10^9$/sec.)

Finally, we would like to comment on two experiments. The first is the vibrating-wire measurement by Carless, Hall, and Hook.[10] In interpretation of their data, the boundary condition $j_s=0$ was employed and found to be quite successful. Note, however, that it does not imply experimental evidence that the surface coefficients, b and c of Eqs. (4), acquire vanishing values, or that the corresponding contribution of the surface entropy production vanishes. This is because of the second boundary condition they employed, $f_0=0$. And as will be shown below, if the rate of heat transfer f_0 vanishes, so does the amplitude of the sq mode, leading to $T'=0$ and therefore $j_s=0$. Let us, for simplicity, think about an infinite, oscillating plane rather than a vibrating wire. Eliminating ΔT from Eqs. (4) and employing $f_0 = -\sigma j_s - \kappa T'/T$, we arrive at

$$Af_0 = Bg' + Cf_0' + DT' + ET'', \tag{10}$$

where $A,B,\ldots,E$ are constants, given by $A=-\rho(c+a/\sigma)/(ab-c^2)$, $B=\beta+\alpha/\rho$, $C=\alpha/s$, $D=\alpha\kappa\rho/(ab-c^2)\sigma T$, and $E=\alpha\kappa/sT$. To zeroth order in ω, the values of g and f_0 at the boundary directly yield the amplitudes of first and second sound, respectively, while Eq. (10) determines the amplitude of the sq mode, proportional to f_0, as given in Eq. (5). (To the given order, only T' and T'' contain the sq amplitude, while both $g'=iq_1 g$ and $f_0'=iq_2 f_0$, representing the damping of first and second sound, respectively, are of first order in ω.) With f_0 being zero, so is the sq amplitude. To the next order in ω, the sq amplitude is proportional to $q_1 Bg$, a tiny quantity of order of the viscous penetration depth over first-sound wavelength. With $f_0=0$, both f_0' and the second-sound contribution to T' are of order ω^2.

The second experiment concerns the U tube already mentioned.[3] Here, things are more complicated. The oscillation in the tube changes both the temperature and the chemical potential of the liquid and forces it to exchange heat and mass with the vapor above. With $f_0, g_0 \neq 0$ this is obviously a situation more general than considered in this paper. Now, one may of course assume appropriate values of the surface transport coefficients to simplify it again, such as putting $b=c=0$. Then $j_s=0$ irrespective of the value of f_0. In fact, in a recent microscopic calculation, this choice was shown to hold for both diffuse and specular reflections at most temperatures.[11] However, we have an observation to report in this context: Instead of putting $b=c=0$, we could have set the entropy $\sigma=0$ to achieve the same results, cf. Eqs. (5). And indeed, the microscopic calculation assumed vanishing entropy from the outset. Because of the low temperature, the entropy in ^{3}He is of course a small quantity, yet what need to be shown are the inequalities $a \gg c\sigma$, $\bar{\kappa} \gg (b-c^2/a)\sigma^2$, and $\bar{\kappa} \gg b\sigma^2 + c\sigma$; cf. Eqs. (5). Hence, the boundary condition $j_s=0$ as the only possible choice still awaits microscopic verification.

In summary, with the help of conservation laws and irreversible thermodynamics, we have derived the boundary condition for a superfluid contained by solid walls. For ^{4}He and vanishing frequencies, they reduce to the much simpler Khalatnikov boundary conditions, which were also shown to yield excellent approximation for finite frequencies. For ^{3}He, the general boundary conditions were applied to various experimentally relevant situations, resulting in a number of predictions including a negative Kapitza resistance and generation of first sound by an oscillating entropy current.

We enjoyed helpful discussions with Mike Cross at the beginning and Peter Wölfe at the end of this work; Nils Schopohl made useful suggestions in between.

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