Crossover in the Anderson Transition: Acoustic Localization with a Flow

S. M. Cohen and J. Machta
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003

and

T. R. Kirkpatrick and C. A. Condat
Department of Physics and Astronomy and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742
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Acoustic waves propagating in a sufficiently disordered medium are localized because of multiple elastic scattering. The presence of a uniform flow in the medium breaks time-reversal invariance and changes the mechanism of localization. A field theory incorporating the crossover induced by a flow field is described and the crossover behavior of the localization length is calculated. An experimental realization of this phenomenon is proposed, using third sound in a superfluid film on a rough substrate with a uniform superfluid flow.

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The qualitative features of the Anderson transition are modified by an external field which breaks time-reversal invariance. Such a perturbation destroys the dominant coherent backscattering mechanism responsible for localization in systems with time-reversal invariance.[1] The result is a new mobility edge and a new set of critical exponents. The purpose of this Letter is to explore the crossover between these two universality classes with particular emphasis on an experimental realization described by the classical wave equation.

In recent years there have been several experimental and theoretical[2,3] studies of localization in disordered classical wave systems. Experiments probing the three-dimensional weak-localization regime have been carried out for light propagation in dense suspensions of polystyrene beads.[4] Experiments have been proposed[5,6] in which one- and two-dimensional localization effects should be observable in third sound propagating on rough substrates. In this Letter we propose an extension of these third-sound experiments in which time-reversal invariance is broken by a superflow.

Near two dimensions the Anderson transition can be described as a phase transition in a nonlinear sigma model with matrix fields.[7-10] In the presence of time-reversal invariance, the action which generates the relevant two-particle Green's function may be expressed by use of real fields and is invariant under pseudo-orthogonal, $O(n,n)$ transformations. When time-reversal invariance is broken, the action must be expressed by means of complex fields and is symmetric under pseudounitary, $U(n,n)$ transformations. These two field theories have distinct nontrivial fixed points, β functions, and exponents. The exponent describing the crossover between these symmetries has been obtained by use of perturbative methods[11,12] and, more recently, by field theory.[13] In this note we describe a field theory which contains both fixed points and present results for the crossover exponent and the crossover function. The field theory which we find is similar to that derived independently in Ref. 13.

Third sound[14] is a propagating long-wavelength disturbance of a superfluid film. At low temperatures third sound has little damping and is well described by the two-dimensional scalar wave equation. On a rough substrate,[15] third-sound waves experience multiple elastic scattering which leads to either the diffusion or localization of the energy of the disturbance. A coarse-grained description of the propagation of third sound at low temperatures on a substrate with random scatterers is given in Ref. 5. It is straightforward to modify this description to include the presence of a uniform flow field with velocity u. The equation of motion for the velocity potential is

$$\{(D/Dt)[1+\chi(r)](D/Dt)-c^2\nabla^2\}\phi(r,t)=0, \qquad (1)$$

where $D/Dt\equiv\partial/\partial t+u\cdot\nabla$ is the convective derivative, $\chi(r)$ is a delta-correlated Gaussian random variable, $\langle\chi(r)\chi(r')\rangle=\gamma^2\delta(r-r')$, and c is the effective sound speed on the substrate. For $u=0$, in one and two dimensions all the modes of (1) are localized with localization lengths that diverge as the frequency or disorder goes to zero. It was shown in Refs. 5 and 6 that the strong-localization regime, where the localization length approaches the mean free path, could be probed by means of third sound.

The central theoretical quantity of interest in the study of Anderson localization is the configuration-averaged two-particle Green's function. This quantity can be generated from a functional integral over two sets of n complex replica fields. After integration over configurations of the Gaussian disorder the averaged two-particle Green's function takes the form

$$\langle G_E^+(r|0)G_E^-(0|r)\rangle = \lim_{n\to 0}\lim_{\eta\to 0^+}\int[d\phi]\phi_{1^+}^*(r)\phi_{1^+}(0)\phi_{1^-}^*(0)\phi_{1^-}(r)\exp\{-A[\phi]\} \tag{2a}$$

with

$$A[\phi] = \int dr \sum_\alpha \phi_\alpha^*(r)s_\alpha[D^2 - s_\alpha\eta + c^2\nabla^2]\phi_\alpha(r) - \frac{\gamma^2}{2}\int dr \sum_{\alpha\beta} s_\alpha s_\beta |D\phi_\alpha(r)|^2 |D\phi_\beta(r)|^2, \tag{2b}$$

where $D \equiv E + iu\cdot\nabla$ and E is the Fourier variable conjugate to t. The replica indices have two components: $\alpha=(a,p)$ with $a=1,2,\ldots,n$ and $p=+,-$; $[d\phi]$ indicates a functional integral over all of the replica fields while $s_\alpha \equiv -ip$ and the $s_\alpha\eta$ term ensures the convergence of the functional integral.

Except for the $s_\alpha\eta$ term the action is invariant under pseudounitary,[16] U(n,n) transformations of the replica fields. Pruisken[17] pointed out that when the time-reversal symmetry-breaking field vanishes ($u\to 0$ in our case) the real and imaginary parts of ϕ_α can be treated as independent replica components and the action has a larger O$(2n,2n)$ symmetry. In the limit $n\to 0$ this is equivalent to the conventional formulation of the time-reversal invariant problem where the action is written in terms of real fields and has an O(n,n) symmetry.

Our object is to construct a field theory which incorporates the orthogonal and unitary symmetries and the crossover between them. To this end we choose real replica fields having three indices, $\alpha=(a,p,\sigma)$ with a and p as before and $\sigma=\uparrow,\downarrow$. The complex fields of (2a) are replaced by two real fields with $\phi_\uparrow$ replacing Reϕ and $\phi_\downarrow$ replacing Imϕ. Complex operators in (2b) are written as 2×2 matrices acting on the spin components of the fields according to the prescription

$$A \to \mathbf{A} = \tau_0\times\mathbf{1}(\mathrm{Re}A) - i\tau_2\times\mathbf{1}(\mathrm{Im}A), \tag{3}$$

where τ_j, $j=1,2,3$, are the Pauli matrices and τ_0 is the 2×2 identity matrix and $\mathbf{1}$ is the $2n\times 2n$ identity matrix; the multiplication signs indicate Kronecker products. In this way the action (2b) is faithfully transcribed into a $4n\times 4n$ real representation. For future reference we note that general $4n\times 4n$ real matrices may be rewritten as $2n\times 2n$ quaternion matrices and that quaternions having only τ_0 and τ_2 components represent complex quantities.

The standard construction[3,7,8] of a field theory in which the Anderson transition can be treated by use of renormalization-group methods proceeds by the introduction of matrix fields which decouple the quartic term in the action:

$$\exp\left\{+\frac{\gamma^2}{2}\int dr[(\mathbf{D}\phi)^T\mathbf{S}\mathbf{D}\phi]^2\right\} = \int[dQ]\exp\left\{-\int dr[\tfrac{1}{2}\,\mathrm{Tr}(\mathbf{Q}^2) - \gamma\phi^T\mathbf{S}^{1/2}\mathbf{DQDS}^{1/2}\phi\right\}, \tag{4}$$

where the functional integral extends formally over the space of real $4n\times 4n$ symmetric matrices. $\mathbf{S}$ is a diagonal matrix having s_α along the diagonal. At this stage it is also convenient to carry out a gauge transformation to eliminate the convective term from the quadratic part of the action.

The ϕ integration can now be done yielding a field theory of interacting matrices having the action

$$A[Q] = \frac{1}{2}\int dr\,\mathrm{Tr}\mathbf{Q}^2 + \tfrac{1}{2}\,\mathrm{tr}\ln\mathbf{C}[Q], \tag{5a}$$

where

$$\mathbf{C}[Q] = E^2\mathbf{1} + c^2\nabla^2\mathbf{1} - \eta\mathbf{S} - \gamma\mathbf{DQD} \tag{5b}$$

and "tr" indicates a trace over both replica and spatial coordinates. The two-particle Green's function is an average of a quadratic form in $\mathbf{Q}$ weighted by $\exp\{-A\}$.

To obtain a tractable field theory the action (5) is expanded about a saddle point to second order in the fluctuations. The saddle point, $\mathbf{Q}_0$, can be chosen diagonal in all replica indices and uniform in space, $\mathbf{Q}_0 \approx \mathbf{S}q_0$. Expanded to second order about $\mathbf{Q}_0$ the action takes the form

$$A[Q] = \mathrm{const} + \frac{1}{2}\int_q B_{\alpha\beta\beta'\alpha'}(k)\,\delta Q_{\alpha\beta}(k)\,\delta Q_{\beta'\alpha'}(-k), \tag{6a}$$

with

$$B_{\alpha\beta\beta'\alpha'}(k) = \delta_{\alpha\alpha'}\delta_{\beta\beta'} - \frac{\gamma^2}{2}\int_q G_{\alpha\alpha'}(q+\tfrac{1}{2}k)\,G_{\beta\beta'}(q-k/2) \tag{6b}$$

where $\int_q = \int dq/(2\pi)^d$, $\delta\mathbf{Q} = \mathbf{Q}-\mathbf{Q}_0$, and $\mathbf{G} \equiv \mathbf{DC}^{-1}\times[\mathbf{Q}_0]\mathbf{D}$.

The coefficient B, defined in (6b), is expanded in powers of k with k-independent terms defining the masses of the Q fields. The Q_{++} and Q_{--} fields have a mass which is finite even when the symmetry-breaking $u\cdot\nabla$ term vanishes. This mass suppresses "longitudinal" fluctuations and to leading order in a cumulant expansion,[18] the Q_{++} and Q_{--} mass terms can be replaced by the constraint that $\mathbf{Q}$ has "transverse" fluctuations only,

$$\mathbf{Q}(r) = \mathbf{U}(r)\mathbf{Q}_0\mathbf{U}^T(r), \tag{7}$$

where $\mathbf{U}(r) = \mathbf{S}^{1/2}\mathbf{T}(r)\mathbf{S}^{-1/2}$ and $\mathbf{T}(r) \in \mathrm{O}(2m,2m)$ for each r.

In the quaternion representation, where $\mathbf{Q} = (1/\sqrt{2})\sum_j \tau_j Q^j$, the Q_{+-}, Q_{-+} fields have massless τ_0 and τ_2 components while their τ_1 and τ_3 components have masses proportional to $(u/c)^2$ for small u/c. The coefficient of the k^2 term in B defines the universe coupling constant. After normalizing the fluctuating fields, $\mathbf{Q} \equiv \delta\mathbf{Q}/q_0$, and retaining terms to order k^2 and $\mathbf{Q}^2$ we obtain the effective action,

$$A[Q] = \frac{1}{2t}\int dr\,\mathrm{Tr}[\nabla\mathbf{Q}\cdot\nabla\mathbf{Q} + g(\mathbf{Q}^2 - \mathbf{Q}\tau_2\mathbf{Q}\tau_2)], \quad (8)$$

subject to the constant (7). Note that $\frac{1}{2}\mathrm{Tr}(\mathbf{Q}^2 - \mathbf{Q}\tau_2\mathbf{Q}\tau_2) = \mathrm{Tr}[(Q^1)^2 + (Q^3)^2]$. Thus the mass term couples only to the τ_1 and τ_3 quaternion components of $\mathbf{Q}$. Higher powers of ∇ and $\mathbf{Q}$ are omitted from (8) and presumed to be irrelevant.

The coefficient in (8) of the massless Q^0 and Q^2 fields is the inverse of the sum of ladder diagrams in a peturbative calculation of the two-particle Green's function. The coefficient of the massive Q^1 and Q^3 fields is the inverse of the sum of maximally crossed diagrams. Thus the mass term corresponds to the cutoff, due to time-reversal symmetry breaking, in the $k \to 0$ divergence of the maximally crossed diagrams.

The mass and coupling constant of the action (8) may be calculated explicitly from (6b) when u/c and $\gamma(E/c)^{d/2}$ are small,

$$t = dE^2\gamma^2/c^2, \quad (9a)$$

$$g/2t = 2u^2 q_0^2/dc^2, \quad (9b)$$

where $q_0 = \gamma\pi S_d E^d/4(2\pi c)^d$.

An anisotropic nonlinear sigma model is obtained by elimination of the Q_{++} and Q_{--} fields by use of the constraint (7). We have carried out a momentum-shell renormalization-group[19-21] (RG) analysis of this model and, for $n=0$, obtain the following flows:

$$dt/d\ln L = -\varepsilon t + t^2/[8\pi(1+g)] + O(t^3), \quad (10a)$$

$$dg/d\ln L = 2g + O(t^2), \quad (10b)$$

where π/L is the rescaled momentum cutoff of the theory.

For $d<2$ there is a single nontrivial fixed point at $t=0$ and the RG flows near the fixed point are unaffected by the velocity field in agreement with the exact one-dimensional results of Condat, Kirkpatrick, and Cohen.[22]

For $d \geq 2$ there are two nontrivial fixed points: (1) $g=0$, $t_c = 8\pi\varepsilon$, and (2) $g=\infty$, $t_c = O(\varepsilon^{1/2})$. The $g=0$ fixed point is clearly identified as the orthogonal fixed point. The $g=\infty$ fixed point is identified by consideration of the quaternion representation of the Q fields. When $g \to \infty$, fluctuations of Q^1 and Q^3 are completely suppressed. The fluctuating Q fields are thus restricted to be complex matrices and the action is identical to that obtained for systems without time-reversal invariance starting with $2n$ complex fields. This action is $\mathrm{U}(n,n)$ invariant and the $g=\infty$ fixed point is the unitary fixed point.

For finite g, the renormalization-group flows describe the crossover from orthogonal to unitary localization. The crossover exponent near the orthogonal fixed point is obtained from the eigenvalues of the RG flow linearized about the orthogonal fixed point. In agreement with Refs. 11-13 we find $\phi = \nu[2 + O(\varepsilon^2)]$ where $\nu = 1/\varepsilon$ is the correlation-length exponent and, for example, ϕ describes the crossover behavior of ξ, the localization length

$$\xi(t,g) = \Delta t^{-\nu} F(g/\Delta t^{\phi}). \quad (11)$$

Here $\Delta t = |t - t_c|$ and all lengths are measured in units of the mean free path, $l_0 \propto (c/E)^{d+1}\gamma^{-2}$. For $d=2$, the behavior of ξ for small t and g can be obtained by integration of the RG flow. The localization length is here defined as the length scale L at which the coupling constant is of order 1.[3] Near the orthogonal fixed point (u/c small) the result is

$$\xi(t,g) = \xi(t,0)F(\xi(t,0)^2 g), \quad (12a)$$

where $\xi(t,0) = \exp(8\pi/t)$ is the orthogonal localization length and the crossover function F is given by

$$F(x) = 1 + x/2 + O(x^2). \quad (12b)$$

For $\xi(t,0)^2 g \gg 1$ the behavior of the localization length is controlled by the unitary fixed point. The β function for the unitary fixed point has been calculated to order t^4 by Hikami[23] from which we obtain $\xi(t, g \gg 1) = \exp[(8\pi/t)^2]$.

For third sound propagating on a rough substrate, both the weak $[\xi(t,0) \gg 1]$ and strong $[\xi(t,0) \approx 1]$ localization regimes are predicted[5] to be experimentally accessible in the absence of a flow. Superfluid films on smooth substrates can sustain steady flow velocities[24] of order $u = 10$ cm/sec and greater. If comparable velocities can be achieved on rough surfaces it will be possible to study the unitary fixed point and the crossover from the orthogonal to the unitary fixed point. Third-sound experiments thus may provide an important test of aspects of localization theory which are difficult to observe in electronic systems because of competing phenomena such as electron-electron interactions, the quantum Hall effect, and the Kondo effect.

In conclusion, we have studied the crossover from orthogonal to unitary localization and found that it is described by an anisotropic nonlinear sigma model with a mass term which couples to two of four quaternion components of the matrix fields. We explicitly carried out the calculation for the case of wave propagation in a disordered medium in which a flow term breaks time-reversal invariance. The method can be applied to other models with time-reversal symmetry breaking such as noninteracting electrons in a magnetic field. However, in this case the mass term in the effective action is more complicated and it is not clear that it is renormalizable.[25]

We propose that experiments be carried out in which third sound is propagated on a rough substrate with a uniform superfluid flow. In such experiments we expect to see a dramatic increase in the localization length from $\xi \sim \exp(E_0/E)^2$ to $\xi \sim \exp(E_0/E)^4$ as the velocity field is turned on. This would be manifest as an increase in the transmission of third sound across a system whose size is of the order of the localization length without a velocity. For small velocities we predict a scaling form for ξ and have calculated the crossover function.

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