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Nonexistence of Small-Amplitude Breather Solutions in ϕ^4 Theory

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For the (1+1)-dimensional Klein-Gordon equation called the ϕ^4 model, there is a known asymptotic series formally representing a "breather" (a real-valued solution that is localized in space and periodic in time) in the limit of small amplitude and frequency just below that of spatially uniform infinitesimal oscillations. We show that even though this expansion is valid to all orders, ϕ^4 theory admits no true breathers in this limit. Instead, what appear in many physical contexts are approximate breathers that slowly radiate their energy to $x = \pm\infty$. We calculate this radiation rate, which lies beyond all orders in the asymptotic expansion.

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Nonlinear Klein-Gordon equations in one space dimension,

$$u_{tt} = u_{xx} - g(u),$$

where $g(0)=0$, $g'(0)>0$, arise in several physical contexts[1-3] ranging from magnetic chains and uniaxial ferroelectrics through nonlinear optics to quantum field theory. One much studied example is the sine-Gordon equation, in which $g(u)=\sin u$. This equation admits exact solutions called breathers that are localized in space and periodic in time[4]:

$$u = 4\tan^{-1}\{(\omega^{-2}-1)^{1/2}\mathrm{sech}(1-\omega^2)^{1/2}x\,\sin\omega t\}.$$

Breathers are important in physical applications,[1,3] and the question of whether other Klein-Gordon equations admit breathers is an important unsolved problem.[5-8]

Another common model is

$$u_{tt} = u_{xx} - 2u + 3u^2 - u^3. \tag{1}$$

With $u=\phi+1$, (1) is called the ϕ^4 model, after the quartic term in its Lagrangean density. The possibility that (1) might admit breathers was suggested both by an asymptotic expansion formally representing such solutions,[1] and by numerical evidence from a spatially discrete ϕ^4 model.[9] Any smooth, real-valued solution of (1) that is periodic in time with frequency $\omega > 0$ has a Fourier representation

$$u(x,t) = \sum a_n(x)\exp(in\omega t), \tag{2}$$

where

$$a_{-n}(x) = a_n^*(x), \tag{3}$$

$$(\partial_x^2 + n^2\omega^2 - 2)a_n = -3\sum_m a_m a_{n-m} + \sum_k\sum_m a_k a_m a_{n-k-m}. \tag{4}$$

The question is, do ω and $\{a_n(x)\}$ exist satisfying (3) and (4) such that $a_n \to 0$ as $x \to \pm\infty$ and the series in (2) converges?

Spatially uniform infinitesimal solutions of (1) oscillate with frequency $\sqrt{2}$, and (1) admits no breathers with $\omega > \sqrt{2}$.[7] For $0 < \omega < \sqrt{2}$, simple analysis indicates that the boundary conditions ($a_n \to 0$ as $x \to \pm\infty$) overdetermine the problem, so that nontrivial breathers are unlikely.[5]

Define $\varepsilon := (2-\omega^2)^{1/2}$ for $\omega < \sqrt{2}$. The asymptotic expansion[1] may be obtained by the assumption of both $0 < \varepsilon \ll 1$ and small amplitudes ($\varepsilon = 0$ will be treated separately). We will show that (4) has no smooth, spatially localized solutions in this limit. Therefore the asymptotic expansion represents no true breathers in the ϕ^4 model, although it represents approximate breathers to all orders.

In this limit, examination of (4) as $x \to -\infty$ [where the $a_n(x)$ vanish] shows that $X := \varepsilon x$ is the natural spatial variable for any possible breather, and that a_1 is the dominant Fourier mode. To show that no true breather exists in this limit, we first drop some of the boundary conditions to define a problem, the unique solution of which must be the breather if one exists. To this end, use (3) to define a_n for $n < 0$. For $n \geq 0$, use boundary conditions as $X \to -\infty$:

$$a_1 \exp(-X) \to C, \quad a_n \to 0, \quad n \neq 1. \tag{5a}$$

Changes of amplitude and phase of the arbitrary complex parameter C correspond respectively to x and t transitions in solutions of (1). One more condition must be imposed to assure that a_0 remains $O(\varepsilon)$ globally; say,

$$|a_0| \leq \tfrac{1}{2}, \text{ for all } X. \tag{5b}$$

Formally, (5) provides enough conditions to make a solution of (4) unique. If (4) and (5) have no solution, then certainly no breather exists in the limit in question. From here on we assume that for small enough $\varepsilon > 0$, (4) and (5) have a unique solution, which we denote by $\{\alpha_n(X;\varepsilon)\}$; we assume that it is analytic in X. (Boundary conditions also dictate uniqueness if $\varepsilon = 0$. But *at* $\varepsilon = 0$ there is no scale, so that no solution can be small.)

Using established methods,[10] one can show that $\{\alpha_n(X;\varepsilon)\}$ has an asymptotic expansion to all orders in powers of ε for which, if C in (5a) is chosen properly, the X dependence occurs only through positive integer powers of $S := \varepsilon \operatorname{sech} \varepsilon X$. The first few terms are

$$\begin{aligned} a_0(X;\varepsilon) &\sim \tfrac{1}{2}(1 + \tfrac{85}{18}\varepsilon^2)S^2 - \tfrac{395}{144}S^4 + O(\varepsilon^6), \\ a_n(X;\varepsilon) &\sim (-S/2\sqrt{6})^n[-2 - 49n\varepsilon^2/18 + (85n/36 + \tfrac{1}{2})S^2 + O(\varepsilon^4)], \quad n > 0. \end{aligned} \tag{6}$$

This expansion is valid for $X \leq R$, for any fixed large R.

If $\{\alpha_n(X;\varepsilon)\}$ actually is a breather, then its spatial reflection is also a breather with the same ε and satisfying the same boundary conditions except for spatial translation. Therefore, it must be spatially symmetric about some appropriate point. We will demonstrate nonexistence of the putative breathers by calculating the asymmetry of $\{\alpha_n(X;\varepsilon)\}$.

It follows from the asymptotic series (6) that for every n, $\partial_X a_n|_0 \sim 0$ to all orders. Thus $\{\alpha_n(X;\varepsilon)\}$ is spatially symmetric to all orders, so that any asymmetry must lie beyond all orders in the asymptotic expansion. Because asymptotic series typically diverge, to go beyond all orders is usually meaningless. However, the asymptotic series for each $\partial_X a_n|_0$ *converges* (trivially, because each term vanishes), so that we may ask for transcendentally small corrections to these series. An essential observation in our method is that it is possible to go beyond all orders in an asymptotic expansion that happens to converge.

Analytically continue into the complex X plane both $\{\alpha_n(X;\varepsilon)\}$ and its asymptotic representation (6). We assume that (4) and (5) have a unique solution, and that the expansion (6) remains asymptotic to it, for $X \leq 0$ except near singularities such as $X = \pm i\pi/2$, where the series ceases to be asymptotic because $|S| \to \infty$. Because to all orders the expansion involves only integer powers of S, both $\mathrm{Im}(a_n)$ and $\mathrm{Re}(\partial_X a_n)$ vanish to all orders in ε along the imaginary X axis between $\pm i\pi/2$. Therefore information beyond all orders is available all along this line segment.

A second essential observation in our approach is that near a singularity like $X = i\pi/2$, where the series loses asymptoticity, one may hope to find an effect that lies beyond all orders on the real axis. To do so, we employ matched asymptotic expansions in the complex X plane.[11]

Define new variables in this "linear region":

$$X =: i\pi/2 + \varepsilon y, \quad a_n(X;\varepsilon) =: A_n(y;\varepsilon), \tag{7}$$

and expand these functions in their own ε expansions. At leading order, the governing equations become

$$(\partial_y^2 + 2n^2 - 2)A_n + 3\textstyle\sum_m A_m A_{n-m} - \sum_k \sum_m A_k A_m A_{n-k-m} = 0. \tag{8}$$

Matching of the inner and outer expansions in an appropriate overlap region, such as $\{\varepsilon y \to 0,\ |y| \to \infty,\ -\pi \leq \arg(y) \leq -\pi/2\}$, yields

$$A_0(y) - \tfrac{1}{2}y^{-2} - \tfrac{395}{144}y^{-4} + O(y^{-6}), \quad A_n(y) \sim \sigma^n[-2 - (85n/36 + \tfrac{1}{2})y^{-2} + O(y^{-4})], \quad n > 0, \tag{9}$$

where $\sigma := (2i\sqrt{6}y)^{-1}$. Equations (8) and (9) are ε independent.

To make (8) well posed, impose (9) as boundary conditions as $\mathrm{Re}(y) \to -\infty$, along with two conditions on $\mathrm{Re}(y) = 0$:

$$\mathrm{Im}(A_0) = 0, \quad \mathrm{Re}(\partial_y A_0) = 0. \tag{10}$$

It is sufficient to impose these conditions on any line, $\mathrm{Im}(y)=\text{const}$, that lies below (i.e., closer to the real X axis than) any singularities of the solution of (8). The unique solution of (8)–(10) must satisfy (9) not only as $\mathrm{Re}(y)\to-\infty$, but also as $\mathrm{Im}(y)\to-\infty$, or any combination thereof.

Let $A_n(y)=:B_n+iC_n$. Note from the infinite series indicated by (9) that the imaginary parts C_n vanish to all orders as $\mathrm{Im}(y)\to-\infty$ along $\mathrm{Re}(y)=0$. Taking the imaginary part of (8) there, one obtains equations for the C_n, with coefficients involving the B_n. Because each C_n is (transcendentally) small, we can linearize these equations to a good approximation. At lowest order (as $|y|\to\infty$), these homogeneous linear equations decouple, and the mth equation has two independent solutions for C_m. The equations are coupled at higher order (as $|y|\to\infty$), so that C_m induces contributions in all the other C_n. For $m\geq 2$

$$C_m\sim v_m\exp[-i(2m^2-2y)^{1/2}],$$
$$C_n\sim\frac{6n}{(2m-1)(m-1)}\sigma^{m-n}C_m,\quad 0\leq n<m,\qquad(11)$$
$$C_n\sim\frac{6n}{(2m+1)(m+1)}\sigma^{n-m}C_m,\quad m<n,$$

where v_m is an arbitrary real constant; the other solution $\{\exp[i(2m^2-2y)^{1/2}]\}$ has been omitted because it grows as $\mathrm{Im}(y)\to-\infty$, so that it violates (9). For $m=0$, C_0 is a linear combination of $\exp(\pm\sqrt{2}y)$ at leading order; neither solution vanishes as $\mathrm{Im}(y)\to-\infty$, and so both are excluded by (9). For $m=1$, the solutions are algebraic at leading order, and they are also excluded by the matching condition (9).

Collecting terms for $\mathrm{Im}(y)\to-\infty$, $\mathrm{Re}(y)=0$, we obtain

$$C_2\sim v_2\exp(-i\sqrt{6}y)[1+O(y^{-1})]+(\sqrt{6}/10iy)v_3\exp(-4iy)+\dots\qquad(12)$$

Similar expressions exist for the other C_n, $n\geq 1$. If $v_2\neq 0$, then its term dominates as $\mathrm{Im}(y)\to-\infty$ in (12) and in the corresponding expressions for every n.

We have estimated v_2 by integrating (8) numerically from $\mathrm{Re}(y)\leq-50$ to $\mathrm{Re}(y)=0$, along lines on which $\mathrm{Im}(y)$ was held fixed, either at -5 or at -6. The equations for A_n ($n\geq 1$) were solved as initial-value problems, with use of three terms in the asymptotic series (9) as initial data, while the A_0 equations were treated as two-point boundary-value problems. We iterated back and forth between the two schemes (usually about a dozen times) until convergence to ten significant figures was obtained. The result is $v_2\cong-(4.5\pm 1.0)\times 10^{-3}$. The precise value of v_2 is of secondary importance, provided that it is not zero.

The last step is to continue (12) back along $\mathrm{Re}(X)=0$ to $X=0$. Because the $\mathrm{Im}(a_n)$ vanish to all orders along the boundary X axis, they satisfy approximately linear equations, with solutions similar to those in (11). A similar analysis here eventually yields results like

$$\mathrm{Im}(a_2)\sim K_1\exp(-i\sqrt{6}X/\varepsilon)+K_2\exp(i\sqrt{6}X/\varepsilon)+O(\varepsilon K_1,\varepsilon K_2),\qquad(13)$$

where K_1 and K_2 are free constants. K_1 is obtained by matching of (13) to (12) as $X\to i\pi/2$; K_2 is obtained by a similar matching as $X\to-i\pi/2$. The final result is that on the real axis at $X=0$, as $\varepsilon\to 0$,

$$\partial_X a_2\sim 2\sqrt{6}v_2\varepsilon^{-1}\exp(-\pi\sqrt{6}/2\varepsilon)\qquad(14)$$

(a_2 is the most asymmetric Fourier mode as $\varepsilon\to 0$). It follows that (4) admits no X-symmetric solutions in the limit $\varepsilon\to 0$, and *a fortiori* that (1) admits no true breathers in this limit. This is our main result.

Next, we compute the radiation rate for an approximate breather in this limit. For each small ε, let $\phi_b(x,t;\varepsilon)$ denote the unique, periodic solution of (1) whose Fourier coefficients satisfy (4) and (5); ϕ_b is asymmetric. Let $\Phi(x,t;\varepsilon)$ denote a symmetric (about $x=0$) solution of (1) with two properties as $x\to-\infty$: (i) $\Phi=\phi_b$ to all orders in ε; and (ii) Φ has no incoming radiation. Then $\phi_r(x,t;\varepsilon):=\Phi-\phi_b$ is exponentially small, so that it satisfies approximately the homogeneous equation obtained by linearization of (1) about ϕ_b, with a radiation condition as $x\to-\infty$ and a nonhomogeneous boundary condition ($\partial_x\phi_r=\partial_x\phi_b$) at $x=0$. These can be solved order by order in ε. At leading order, because a_2 is the most asymmetric Fourier coefficient, one finds

$$\phi_r(x,t;\varepsilon)=4v_2\exp(-\pi\sqrt{6}/2\varepsilon)[\sin(\sqrt{6}x+2\omega t)+O(\varepsilon)].\qquad(15)$$

This is also the dominant behavior of $\Phi(x,t;\varepsilon)$ as $x\to-\infty$, because ϕ_b vanishes there.

Because $\partial_x\Phi|_0=0$, conservation of energy yields

$$\partial_t\frac{1}{2}\int_{-L}^{0}dx\{(\partial_t\Phi)^2+(\partial_x\Phi)^2+\tfrac{1}{2}(\Phi^2-1)^2\}=-[\partial_t\Phi\,\partial_x\Phi]_{x=-L}.$$

If $L\varepsilon\gg 1$, the energy flux at $x=-L$ can be estimated from (15). The result is the leading-order approximation of the (time-averaged) radiation rate of an approximate breather of small amplitude:

$$\partial_t(H/2) = -8\sqrt{3}v_2\exp(-\pi\sqrt{6}/2\varepsilon)[1+O(\varepsilon^2)]. \quad (16)$$

One shows from this that as $t \to \infty$, $H \sim \pi\sqrt{6}/2\ln t$.

In conclusion, we mention specifically two questions of current interest in physics upon which these results shed light. First, in applying (1) to problems in condensed matter at finite temperature, one would expect to see long-lived breatherlike objects along with radiation (phonons) at the appropriate frequencies, predicted by (15) and its generalizations. Second, the extremely slow decay of these approximate breathers suggests that even though these solutions are neither strictly periodic nor localized, they still may be relevant for determination of the energy spectrum of the quantized ϕ^4 theory, via an approximate semiclassical quantization.[1,12]

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