Algebraic Description of the Skyrmion and Its Quantization for Finite N

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We present a bosonic realization of the SU(2)⊗SU(2) algebra of the skyrmion. By imbedding the algebra in U(4), we introduce an additional quantum number N, which we identify with the number of colors, N_c. We show that the skyrmion is a coherent state of the U(4) algebra in the large-N limit and generalize that state to finite N. For $N=N_c=3$, we recover the SU(4) quark model. The algebraic $1/N_c$ corrections to one-body matrix elements in the skyrmion are discussed.

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The skyrmion offers an attractive picture of the nucleon as part of a classical solution of a nonlinear field theory of chiral mesons that arises from QCD in the large-N_c limit.[1] The nucleon and Δ can be obtained from the skyrmion by use of semiclassical quantization methods to project the spin S and isospin I, giving a tower of $I=S$ states with $I=S=\frac{1}{2},\frac{3}{2},\ldots$.[2] The unbounded nature of this tower is a manifestation of the large-N_c limit. Much work has been done in the study of the projections, and other interesting features of the model.[3] Most of this work exploits the underlying SU(2)⊗SU(2) structure of the skyrmion algebra rather than the nonlinear nature of the field theory.

On a different front, it has recently been shown in nuclear and molecular physics that dynamical symmetries are conveniently expressed in terms of interacting bosons. These models applied to nuclear physics [the interacting-boson model based on U(6)][4] and to molecular physics [the vibron model based on U(4)][5] have been very successful at correlating a great deal of data both in structure and scattering and in providing a simple and elegant method for dealing with complex systems.

In this paper we present a realization of the skyrmion algebra in terms of interacting bosons. We show that the skyrmion is a coherent state of the U(4) algebra in the large-N limit, N being the number of bosons. We identify N with N_c, the number of colors in QCD. This permits a simple generalization of the skyrmion to finite N_c, makes the projection calculations very direct, and offers considerable scope for generalization to flavor SU(3) and to the meson-baryon sector.

We begin by noting the isomorphism SU(2)⊗SU(2) ≃O(4). The algebra of O(4) is expressed by the commutation relations among two three-component operators K_i and D_i $(i=1,2,3)$, i.e., $[K_i,K_j]=i\varepsilon_{ijk}K_k$; $[K_i,D_j]=i\varepsilon_{ijk}D_k$; and $[D_i,D_j]=i\varepsilon_{ijk}K_k$, with Casimir invariants $\sum_i(K_i^2+D_i^2)$ and $\sum_i K_iD_i$. Alternatively, the operators $S_i=\frac{1}{2}(K_i+D_i)$ and $I_i=\frac{1}{2}(K_i-D_i)$ generate SU(2)⊗SU(2). This algebra is easily realized in terms of the a's of Adkins, Nappi, and Witten[2] (ANW) by $K_i=-i\varepsilon_{ijk}a_j\partial/\partial a_k$ and $D_i=i(a_i\partial/\partial a_4-a_4\partial/\partial a_i)$. These a's are related to the SU(2) unitary rotations, A, of the skyrmion by $A=a_4+ia_i\tau_i$ with $\sum_{i=1}^4 a_i^2=1$, so that S_i and I_i become the spin and isospin, respectively. The boson realization of the algebra is given in terms of four boson operators, b_i $(i=1,2,3,4)$, by

$$b_i=(a_i+\partial/\partial a_i)\sqrt{2},$$
$$b_i^\dagger=(a_i-\partial/\partial a_i)/\sqrt{2}, \qquad (1)$$

satisfying $[b_i,b_j^\dagger]=\delta_{ij}$, so that $K_i=-i\varepsilon_{ijk}b_j^\dagger b_k$, and $D_i=i(b_i^\dagger b_4-b_4^\dagger b_i)$. Also in this realization, $S^2-I^2=\sum_{i=1}^3 K_iD_i$ is zero [symmetric representations of O(4)] so that states generated by the algebra will have $I=S$. The constraint $A^\dagger A=1$ (or equivalently $\sum_i^4 a_i^2=1$) is a condition on the Hilbert space implemented by our taking O(4) eigenstates.

One can imbed O(4) in U(4) which will then yield representations with fixed boson number, $N=\sum_{i=1}^4 b_i^\dagger b_i$. In the boson realization both the spin and isospin operators and therefore also the Hamiltonian $H=M+\mathbf{S}^2/2\mathcal{J}$ conserve the number of bosons. We review some properties of the group chain U(4)⊃O(4)⊃O(3)⊃O(2). In the symmetric representation, the states are $|[N],\sigma,K,M\rangle$ with allowed values $\sigma=N,N-2,\ldots,(1\text{ or }0)$; $K=0,1,\ldots,\sigma$; $K_3=M=-K,-K+1,\ldots,K$. K_i and D_i commute with N. In the boson realization of the algebra the U(4) states are Nth-order polynomials in $b_i^\dagger$,[6]

$$|[N]\sigma KM\rangle=B_\sigma^N\sum_{n=0}^{[(\sigma-K)/2]}F_n(\sigma,K)(b_4^\dagger)^{\sigma-K-2n}\left(\sum_{i=1}^4 b_i^\dagger b_i^\dagger\right)^{(N-\sigma)/2+n}[4\pi/(2K+1)!!]^{1/2}i^K\mathcal{Y}_{KM}(b_1^\dagger,b_2^\dagger,b_3^\dagger)|0\rangle, \qquad (2)$$

with B_σ^N, $F_n(\sigma,K)$, and $\mathcal{Y}_{KM}$ defined in Ref. 6, Eqs. (4.21), (4.23), and (4.10), respectively.

The connection between these states and the states of good spin and isospin is made by an ordinary Clebsch-Gordan coefficient. Using the fact that in symmetric representations $S=I=\sigma/2$ and $K_i=I_i+S_i$, we have for the states of good I and S

$$|[N],I=S=\sigma/2,I_3S_3\rangle=\sum_{K,M}|[N]\sigma KM\rangle\langle II_3SS_3|KM\rangle. \tag{3}$$

If N is even these will contain states of $I=S=0,1,\ldots,N/2$, while if N is odd these will contain states of $I=S=\frac{1}{2}$, $\frac{3}{2},\ldots,N/2$. This suggests that we can identify N with N_c, the number of colors. In the large-N limit, we have an infinite tower of $I=S$ states as in the skyrmion, while for $N=N_c=3$ we have $I=S=\frac{1}{2}$ (nucleon) and $\frac{3}{2}$ (Δ). Combining (2) and (3), we can construct states of good spin and isospin. For example, for the proton ($\sigma=1$) with spin up and general N odd, we have

$$|[N],I=S,I_3S_3\rangle=|[N]\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}\rangle=\frac{-i}{\sqrt{2}}B_1^N(b_1^\dagger+ib_2^\dagger)\left(\sum_{i=1}^4 b_i^\dagger b_i^\dagger\right)^{(N-1)/2}|0\rangle. \tag{4}$$

The above discussion suggests that the skyrmion corresponds to a U(4) coherent state in the large-N limit (with N odd). To show this correspondence explicitly, we study how the spin-isospin projection functions are constructed in our algebra and how they correspond to the functions discussed by ANW, and show that for large N the expectation value of operators is the same as in the skyrmion case.

The coherent state[7] is known to be useful in studying the connection between algebraic and geometric models[8]:

$$|[N]\beta_i\rangle=\frac{1}{\sqrt{N!}}\left(\sum_{i=1}^4\beta_i b_i^\dagger\right)^N|0\rangle, \tag{5}$$

with $\sum_i\beta_i^*\beta_i=1$. It is convenient to parametrize the Euler-Rodrigues parameters β_i in terms of $\Omega=(\chi,\theta,\phi)$, i.e., $\beta_1=\sin\chi\sin\theta\cos\phi$; $\beta_2=\sin\chi\sin\theta\sin\phi$; $\beta_3=\sin\chi\cos\theta$; and $\beta_4=\cos\chi$. Taking the overlap of Eqs. (2) and (5), we obtain

$$\begin{aligned}&\langle[N]\Omega\,|\,[N]\sigma KM\rangle=P_\sigma^N g_{\sigma KM}(\Omega),\\ &g_{\sigma KM}(\Omega)=Q_K^\sigma C_{\sigma-K}^{K+1}(\cos\chi)(i\sin\chi)^K Y_{KM}(\theta,\phi),\\ &P_\sigma^N=[4\pi^2N!/(N-\sigma)!!(N+\sigma+2)!!]^{1/2},\\ &Q_K^\sigma=2^KK![2(\sigma+1)(\sigma-K)!/\pi(\sigma+J+1)!]^{1/2},\end{aligned} \tag{6}$$

where $C_{\sigma-K}^{K+1}$ is a Gegenbauer polynomial and P_σ^N is defined so that $g_{\sigma KM}(\Omega)$, which is independent of N, is orthonormal with respect to the measure of the three-sphere.

The quantized skyrmion wave functions of ANW are now given by

$$\Psi_{I=S,I_3,S_3}(\Omega)=\sum_{KM}\langle II_3SS_3|KM\rangle g_{\sigma=2I,K,M}(\Omega), \tag{7}$$

where the a's in ANW correspond to the $\beta(\Omega)$'s. The functions $g_{\sigma KM}$ are either even or odd under the "parity" transform, $\Omega\to(-\Omega)=(\chi+\pi,\theta,\phi)$ or equivalently $(\pi-\chi,\pi-\theta,\phi+\pi)$: $g_{\sigma KM}(-\Omega)=(-)^\sigma g_{\sigma KM}(\Omega)$. In the limit of N large and odd (and therefore also σ odd), $\{g_{\sigma KM}(\Omega)\}$ becomes complete in the parity-odd function space. Thus the N large and odd coherent state contains an infinite tower of $I=S=$half-integer states. The coherent state with N large and even corresponds to an integer-spin skyrmion. The existence of this alternative is well known (see Ref. 2). Henceforth, we only discuss the case of physical interest—N odd.

To complete the connection with the skyrmion it can be shown that the matrix element of any k-body boson operator (k finite) becomes diagonal in the coherent-state basis in the large-N limit. With the classical limit[8] of a k-body operator $\hat{O}_k$ defined by

$$\mathcal{O}_k(\Omega)\equiv\lim_{N\to\infty}\langle[N]\Omega\,|\,\hat{O}_k\,|\,[N]\Omega\rangle/N^k, \tag{8}$$

we obtain in the large-N limit

$$\lim_{N\to\infty}\langle[N],I=S,I_3S_3\,|\,\hat{O}_k\,|\,[N],I'=S',I_3'S_3'\rangle/N^k=\int d\Omega\,\Psi^*_{I=S,I_3,S_3}(\Omega)\mathcal{O}_k(\Omega)\Psi_{I'=S',I_3',S_3'}(\Omega), \tag{9}$$

which coincides with ANW's formula for matrix elements.

By imbedding the spin-isospin group into U(4), we have introduced an additional quantum number N, which we identify as the number of colors N_c. This identification enables us to compute the dependence of physical quantities, such as the g_A factor, magnetic moments, and transitions, on the number of colors. Recall that the classical soliton U_0 is quantized by the isospin rotation $A=a_4+ia_i\tau_i$ ($\sum_i a_i^2=1$) (in the notation of ANW). The corresponding element of the orthogonal space rotation group is given by $R_{ij}=\frac{1}{2}\mathrm{Tr}[\tau_iA\tau_jA^\dagger]$. By using Eq. (1), we write the R_{ij}'s in terms of the boson operators: for instance,

$$\begin{aligned}&R_{33}=a_4^2+a_3^2-a_1^2-a_2^2=R_{33}^c+R_{33}',\\ &R_{33}^c=b_4^\dagger b_4+b_3^\dagger b_3-b_1^\dagger b_1-b_2^\dagger b_2,\end{aligned} \tag{10}$$

where R_{ij}^c conserves boson number, while R_{ij}' is a term which changes the boson number by either 2 or -2. Since all states are characterized by a fixed boson number N, the matrix elements of R'_{ij} vanish. The R^c_π's with the spin S and the isospin I form an SU(4) algebra,

$$[I_p, R^c_{qj}/2] = i\varepsilon_{pqr} R^c_{rj}/2,$$
$$[S_i, R^c_{qj}/2] = i\varepsilon_{ijk} r^c_{qk}/2, \quad (11)$$
$$[R^c_{pi}/2, R^c_{qj}/2] = i(\delta_{ij}\varepsilon_{pqr} I_r + \delta_{pq}\varepsilon_{ijk} s_k).$$

Therefore the isovector axial vector current is given by $A^p_i = -R^c_{pi}/2$, which is consistent with the Noether current for the skyrmion (ANW). From Eq. (11) we observe that p is an isospin index and i is a spin index in R_{pi}. Note that the last commutation relation of Eq. (11) holds only for the part R^c_{ij} which conserves boson number, while the full R_{ij}'s commute with each other. It has been pointed out by several authors[9] that the skyrmion has the same symmetry structure as the large-N_c limit of the quark model. In Eq. (11) we have shown that even for finite N the SU(4) current algebra can be obtained by restricting the operators to be boson-number conserving. This was to be expected since our U(4) algebra is given by the fifteen SU(4) generators and the number operator N. For $N = N_c = 3$, the U(4) irreducible representations are identical to those of the SU(4) quark model. Generalization of this $SU(4) \subset U(4)$ to $SU(6) \subset U(6)$ should be straightforward.

We now want to study the N dependence of one-body operators (bilinear forms of the bosons) to establish that in the large-N limit we obtain the skyrmion results, while for $N = N_c = 3$ we obtain the quark-model results. Of the sixteen U(4) generators, which are one-body operators in the boson realization, i.e., $b_i^\dagger b_j$ $(i,j = 1,\ldots,4)$, the six $SU(2)\otimes SU(2)$ generators, I_p's and S_i's, have matrix elements independent of N. The number operator N has a trivial N dependence. We calculate the N dependence of the matrix element of the remaining nine one-body operators, R^c_{pi}'s: The diagonal matrix element is

$$\langle [N], I=S, I_3S_3 \,|\, R^c_{pi} \,|\, [N], I=S, I'_3S'_3\rangle = -Nf(N,I)\langle 4I_pS_i\rangle, \quad (12)$$

where $f(N,I) = [I/(I+1)](1+2/N) = \frac{1}{3}\times\frac{5}{3}$ $(N=3)$ and $\frac{1}{3}$ (large N) for the nucleon $(I=\frac{1}{2})$. For $N = N_c = 3$ this result is in agreement with the quark model. For large N it reproduces the skyrmion result recalling the definition of the classical operator (8). Therefore the $1/N$ correction to the nucleon g_A factor in the skyrmion is given by $1+2/N$. The isovector magnetic moment is also proportional to R^c_{pi}: $\mu_{pi} = -\mu_0 R^c_{pi}$,[2] and thus has the same $1/N$ correction. Because the isoscalar magnetic-moment operator is proportional to the spin S_i, it is independent of N. For the skyrmion, $\mu_0 = \mathcal{I}/2$, $\mathcal{I}$ being the moment of inertia of the skyrmion. It is well known that the isovector magnetic moment and the g_A factor of the nucleon are too small when calculated in the Skyrme model. The enhancement factor $\frac{5}{3}$ for $N = N_c = 3$ gives a natural remedy for this discrepancy.[10] The magnetic transition matrix element between the nucleon and Δ is

$$\langle [N], I=S=\tfrac{3}{2} \,|\, \mu_{33} \,|\, [N], I=S=\tfrac{1}{2}\rangle = \mu_0 N[(1-1/N)(1+5/N)/2]^{1/2}\langle S^t_3 I^t_3\rangle, \quad (13)$$

where S^t_3 (I^t_3) is the transition spin (isospin) operator normalized by $\langle S=S_3=\frac{3}{2} \,|\, S^t_{+1} \,|\, S=S_3=\frac{1}{2}\rangle = 1$. The matrix element is zero if $N=1$ because no $I=S=\frac{3}{2}$ state exists then. Again there is an enhancement factor of $\frac{4}{3}$ between $N=3$ and the large-N limit, which is necessary for agreement with experiment. The transition matrix element of the quadrupole operator is proportional to that of R^c_{33}, and therefore the $E2/M1$ ratio for Δ photoproduction does not depend on N.

In conclusion, we have seen that by use of a boson realization of $SU(2)\otimes SU(2) \simeq O(4)$ and imbedding of the O(4) in U(4), the skyrmion can be thought of as a U(4) coherent state in the classical limit (large N). This makes the projections of states of good spin and isospin very simple and easily permits generalizations to finite N. We have identified N (number of bosons) with N_c (number of colors) and have studied the N_c dependence of some matrix elements. This algebraic leading N_c correction to the skyrmion gives a significant effect, although it is not the only $1/N_c$ correction. These corrections have been discussed by several authors.[9,10] We have found that for $N = N_c = 3$ the SU(4) quark-model results are recovered in our formalism. Our approach points to obvious generalization to $SU(6) \supset SU(3)_{\text{flavor}} \otimes SU(2)$ and to new ways to approach the meson-nucleon[11] and nucleon-nucleon problems.[12]

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