Quantum-Mechanical Treatment of the Skyrme Lagrangean, and a New Mass Term

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We consider the quantum mechanics of the SU(2) skyrmion model in the framework of collective-coordinate quantization. We treat the Lagrangean quantum-mechanically from the beginning. A new mass term with negative sign appears, which may play an important role in stabilizing the rotating chiral soliton.

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The purpose of the present note is to reexamine quantum structures of the SU(2) Skyrme Lagrangean, especially of the skyrmion mass term, in the framework of the collective-coordinate formalism.[1] Usually, as in the standard approach,[1] canonical quantization is performed only after the Lagrangean is expressed concretely in terms of the collective coordinate treated classically. In order to go to a quantum theory from a classical one, it is necessary to specify the quantization procedure. In contrast to the procedure of Adkins *et al.*,[1] we treat the Skyrme Lagrangean quantum mechanically from the beginning in accordance with the quantization procedure of nonlinear theories, which has already been investigated by some authors.[2,3] [The SU(2) Skyrme model is the simplest example of quantum mechanics on a curved space.] In our treatment, we have to take care of the ordering from the outset. It is expected that some new terms appear. In the following, it is to be pointed out that, in the skyrmion mass, a new negative term appears, which serves to stabilize the rotating chiral solitons; therefore, the instability problem[4] of such solitons should be reconsidered with the existence of the new term taken into account.

We start with the SU(2) Skyrme Lagrangean

$$L(U_{L\rho};\mathbf{x},t)=f_\pi^2\,\mathrm{Tr}(U_{L\rho}U_{L\rho})/4+\mathrm{Tr}([U_{L\rho},U_{L\lambda}]^2)/32e_s^2+f_\pi^2 m_\pi^2\,\mathrm{Tr}(U+U^\dagger-2)/4, \tag{1}$$

with $U_{L\rho}=(\partial_\rho U)U^\dagger$ and $(f_\pi)_{\rm expt}\simeq 93$ MeV. Existence of the soliton solution with appropriate boundary conditions is assumed, and the collective coordinate $A(t)$ is introduced as in Ref. 1: $U(\mathbf{x},t)=A(t)\sigma(\mathbf{x})A(t)^\dagger$. We use a set of three real parameters q^b's $(b=1,2,3)$ so as to specify an SU(2) matrix $A(t)$. As $A^\dagger(\partial A/\partial q^a)\equiv A^\dagger\partial_a A$ belongs to the Lie algebra of SU(2), we can write

$$A^\dagger\partial_a A=i\tau_B C(q)_a{}^B/2. \tag{2}$$

Here the summation convention is adopted. The inverse of $(C_b{}^B)$ is defined as

$$C^b{}_E C_b{}^D=\delta_E{}^D,\quad C^b{}_E C_d{}^E=\delta_d{}^b. \tag{3}$$

$\{C_b{}^D\}$ and $\{C^b{}_D\}$ have various important properties, one of which is

$$C^b{}_E\partial_b C^a{}_B-C^b{}_B\partial_b C^a{}_E=-\varepsilon_{EBF}C^a{}_F, \tag{4}$$

where ε_{EBF} is the totally antisymmetric tensor.

The basic assumption in the construction of quantum mechanics of the present model is that we require the commutation relation between $\dot q^d\equiv dq^d/dt$ and q^b:

$$[\dot q^d,q^b]=-if^{db}(q), \tag{5}$$

where $f^{ab}(q)$ is a function of only q's and is determined after the quantization condition is imposed. Next we define

$$w^B=\{\dot q^a,C_a{}^B\}/2, \tag{6}$$

and also the quantum form of $\dot A\equiv dA/dt$ as

$$\dot A(q)=\{\dot q^a,\partial_a A(q)\}/2. \tag{7}$$

By employment of

$$[\dot q^a,A(q)]=-if^{ab}\partial_b A(q), \tag{8}$$

it is easy to derive

$$A^\dagger\dot A=i\tau_B w^B/2+if^{BB}/8, \tag{9a}$$

where

$$f^{BD}=C_a{}^B C_b{}^D f^{ab}. \tag{9b}$$

Note that $\dot A$, as defined above, has the desired property:

$$A^\dagger\dot A+\dot A^\dagger A=\dot A A^\dagger+A\dot A^\dagger=0. \tag{10}$$

The quantum form of $A^\dagger\dot A$ has a term proportional to the unit matrix. This corresponds to a term $a_0\dot a_0+a_B\dot a_B$ appearing in $A^\dagger\dot A$ when we use (a_0,a_E) variables with

$A=a_0+ia_B\tau_B$.[1] We can, however, take $A^\dagger\dot{A}$ to be equal effectively to $i\tau_B w^B/2$ in $L(U_{L\rho})$, because

$$U_{L4}=-iA[A^\dagger\dot{A}-\sigma A^\dagger\dot{A}\sigma]A^\dagger=X_{BD}A\tau_D w^B A^\dagger, \tag{11a}$$

where

$$\tau_B-\sigma(\mathbf{x})\tau_B\sigma(\mathbf{x})^\dagger=2X(\mathbf{x})_{BD}\tau_D. \tag{11b}$$

With the help of

$$[w^B,A]=f^{BD}A\tau_D/2, \tag{12}$$

we can demonstrate that

$$L(U_{L\rho};\mathbf{x},t)=a(\sigma;\mathbf{x})_{BD}w^Bw^D/2+[\text{term of order }(w^B)^0], \tag{13}$$

where

$$a(\sigma;\mathbf{x})_{BE}=4[Y^{(1)}_{BE,DD}+Y^{(2)}_{BE,DD}],\quad Y^{(1)}_{BE,DK}=f_\pi^2X_{BD}X_{EK}/4,\quad Y^{(2)}_{BE,DK}=\xi_k{}^F\xi_k{}^HX_{BG}X_{EL}\varepsilon_{FGD}\varepsilon_{HLK}/16e_s^2, \tag{14a,b,c}$$

$$[\partial\sigma(\mathbf{x})/\partial x^k]\sigma(\mathbf{x})^\dagger=i\tau_B\xi(x)_k{}^B/2. \tag{14d}$$

The proof is given as follows: First, the contribution of U_{L4} to $L(U_{L\rho})$,

$$L(U_{L4})\equiv f_\pi^2\,\mathrm{Tr}(U_{L4}U_{L4})/4+\mathrm{Tr}([U_{Lk},U_{L4}][U_{Lk},U_{L4}])/16e_s^2, \tag{15a}$$

is rewritten as

$$L(U_{L4})=(Y^{(1)}_{BE,KJ}+Y^{(2)}_{BE,KJ})\mathrm{Tr}(Aw^Bw^E\tau_K\tau_JA^\dagger). \tag{15b}$$

Because of (12), we cannot simply eliminate A and $A^\dagger$ in the trace. This part is expressed as

$$\mathrm{Tr}(\cdots)=2w^Bw^E\delta_{KJ}-i(f^{BM}w^E+w^Bf^{EM})\varepsilon_{MKJ}+f^{BM}f^{EN}(\delta_{MK}\delta_{JN}-\delta_{MJ}\delta_{KN}+\delta_{MN}\delta_{KJ})/2. \tag{16}$$

The first term on the right-hand side of (16) corresponds to the term obtained in the standard approach[1] and leads to the rotational energy, while the remaining terms are new contributions which are brought about by our quantization procedure. Making use of

$$Y^{(i)}_{BE,KL}(\mathbf{x})=Y^{(i)}_{EB,LK}(\mathbf{x}),\quad i=1,2, \tag{17a}$$

we see that the terms of order $(w^B)^1$ on the right-hand side of (16) reduce to ones of order $(w^B)^0$ because of

$$[w^B,f^{EM}(q)]=-if^{bd}C_b{}^B\partial_df^{EM}. \tag{17b}$$

Similarly, we can prove from (13)

$$L(U_{L\rho};\mathbf{x},t)=\dot{q}^ad(\sigma;\mathbf{x})_{ab}\dot{q}^b/2+[\text{term of order }(\dot{q}^a)^0], \tag{18}$$

where $d(\sigma;\mathbf{x})_{ab}\equiv a(\sigma;\mathbf{x})_{BD}C_a{}^BC_b{}^D$.

Thus, we have

$$L(U_{L\rho})\equiv\int d^3x\,L(U_{L\rho};\mathbf{x},t)=\Lambda(\sigma)w^Bw^B/2+[\text{term of order }(w^B)^0] \tag{19a}$$
$$=\dot{q}^ag_{ab}(q)\dot{q}^b/2+[\text{term of order }(q^d)^0],$$

where

$$\Lambda(\sigma)\delta_{BE}\equiv\int d^3x\,a(\sigma;\mathbf{x})_{BE}, \tag{19b}$$

$$g_{ab}(q)\equiv\Lambda(\sigma)C_a{}^BC_b{}^B. \tag{19c}$$

Now, we can define the canonical momentum p_a, conjugate to q^a, as

$$p_a\equiv\partial L(U_{L\rho})/\partial\dot{q}^a=\tfrac{1}{2}\{\dot{q}^b,g_{ab}\}. \tag{20}$$

We impose the commutation relations

$$[p_a,q^b]=-i\delta_a{}^b,\quad\text{others}=0. \tag{21}$$

Then we easily obtain

$$f^{ab}g_{ad}=\delta_d{}^b, \tag{22a}$$

$$A^\dagger\dot{A}=i\tau_Bw^B/2+3i/8\Lambda(\sigma), \tag{22b}$$

$$[w^B,A]=A\tau_B/2\Lambda(\sigma). \tag{22c}$$

For $R_B\equiv-\{p_a,C^a{}_B\}/2$, we can prove from (4)

$$[R_B,R_D]=-i\varepsilon_{BDE}R_E, \tag{23a}$$

$$R_B=-w^B\Lambda(\sigma). \tag{23b}$$

Using these relations, we get

$$L(U_{L\rho})=R_BR_B/2\Lambda(\sigma)-[M(\sigma)+\Delta M(\sigma)], \tag{24}$$

where $\Lambda(\sigma)$ and the "classical" mass term $M(\sigma)$ reduce to $\Lambda[F]$ and $M[F]$[1,4] for the hedgehog form of

$\sigma(\mathbf{x}) = \exp[iF(r)\hat{\mathbf{x}}\cdot\boldsymbol{\tau}]$,

$$\Lambda[F] = \frac{8\pi}{3}\int_0^\infty dr\, r^2 s^2 \left\{ f_\pi^2 + \frac{1}{e_s^2}\left[F'^2 + \frac{s^2}{r^2} \right] \right\}, \tag{25a}$$

$$M[F] = 2\pi \int_0^\infty dr\, r^2 \left\{ f_\pi^2 \left[F'^2 + \frac{2s^2}{r^2} \right] + \frac{s^2}{r^2}\frac{1}{e_s^2}\left[2F'^2 + \frac{s^2}{r^2} \right] + 2m_\pi^2 f_\pi^2 (1-c) \right\}. \tag{25b}$$

$\Delta M(\sigma)$ is the new contribution appearing through our quantum-mechanical treatment of the Lagrangean part with the time derivative, (15a); in other words, this contribution comes from the last part on the right-hand side of (16). Note that the second part on the right-hand side of (16) vanishes as a result of $f^{BD} = \delta_{BD}/\Lambda(\sigma)$. $\Delta M(\sigma)$ for the hedgehog solution is given by

$$\Delta M[F] = \frac{-2\pi}{\Lambda[F]^2}\int_0^\infty dr\, r^2 s^2 \left\{ f_\pi^2 + \frac{1}{2e_s^2}\left[2F'^2 + \frac{s^2}{r^2} \right] \right\} = \frac{-3}{4\Lambda[F]} + \frac{1}{\Lambda[F]^2}\frac{\pi}{e_s^2}\int_0^\infty dr\, s^4. \tag{26}$$

For the hedgehog configuration, the energy of $I^2 = J^2 = l(l+1)$ state is from (24) equal to

$$H_l[F] = M[F] + \Delta M[F] + l(l+1)/2\Lambda[F]. \tag{27}$$

Note that both the second and the third terms on the right-hand side are of the order $\hbar^2$. The integrodifferential equation is derived[4] so as to minimize H_l with respect to F. With the aim of examining the asymptotic solution, we derive the linear differential equation for sufficiently large r:

$$r^2F'' + 2rF' - 2F - \mu_l^2 r^2 F = 0, \tag{28a}$$

with

$$\mu_l^2 = m_\pi^2 + \frac{1}{\Lambda[F]^2}\left\{ -\frac{2l(l+1)}{3} + 1 - \frac{8\pi}{3e_s^2\Lambda[F]}\int_0^\infty dr\, s^4 \right\}. \tag{28b}$$

$\mu_l^2 > 0$ is needed for a desirable asymptotic behavior $F(r) \sim e^{-\mu_l r}/r$. We have for the chiral limit

$$\mu_{l/2}^2(m_\pi \to 0) = \frac{1}{\Lambda^3}\frac{4\pi}{3f_\pi e_s^3} J[F], \tag{29}$$

with

$$J[F] \equiv \int_0^\infty dz\, z^2 \tilde{s}^2 (1 + \tilde{F}'^2 - \tilde{s}^2/z^2),$$

$$z = f_\pi e_s r, \quad \tilde{F}(z) = F(r), \quad \tilde{s} = \sin\tilde{F}.$$

$J[F]$ is not always negative as a result of ΔM and possibly becomes positive. It is necessary to solve the integrodifferential equation to see whether or not there exists a physically acceptable $F(r)$ with $J[F] > 0$, but this goes beyond the scope of the present paper. Nevertheless, in spite of the existence of ΔM, the instability for the $l = \frac{1}{2}$ state still remains for a small m_π. From the above consideration, we see that the argument given by Braaten and Ralston[4] concerning the Δ-N mass difference does not hold straightforwardly.

We add lastly one remark on the ambiguity of the starting Lagrangean form. $L(U_{R\rho})$, obtained from $L(U_{L\rho})$ by substitution of $U_{R\rho} \equiv U^\dagger \partial_\rho U$ for $U_{L\rho}$, is shown to be equal to $L(U_{L\rho})$ quantum mechanically. So there is no ambiguity in the form of ΔM.

Details of the present paper and considerations of some effects brought about by our quantum-mechanical treatment will appear elsewhere.

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