Closed Strings in Open-String Field Theory

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An explicit construction of closed-string states is given in terms of open-string oscillators acting on the open-string Fock vacuum. The closed-open-open coupling is computed for the graviton and agrees with the standard result. The closed string states are not annihilated by the Becchi-Rouet-Stora-Tyutin operator Q. Instead they generate inner derivations which anticommute with Q. This is sufficient to insure gauge invariance of tree-level S-matrix elements. It also implies that the closed-string states obey, level by level, the linearized equation of motion of the cubic action.

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Recently it has been conjectured[1] that open-string scattering amplitudes computed in Witten's *open-string* field theory[2] contain all the *closed-string* contributions required by Lorentz invariance and unitarity. Closed strings arise because certain one-loop open-string diagrams can be cut in a manner that produces a closed-string pole. For example, one of the one-loop four-gluon scattering diagrams representing two-gluon exchange can be continuously deformed into a "tree" diagram with an internal graviton. These processes are represented by separate diagrams in light-cone string field theory, but this is possible only because the Feynman rules are explicitly not Lorentz invariant. Thus, it is hard to see how a Lorentz-invariant open-string field theory can avoid containing closed strings.

This raises a number of puzzles. For example, if closed-string poles appear in scattering amplitudes, it seems that unitarity requires that they also appear as external states. One cannot remedy this by adding an explicit closed-string field to the theory (as in the non-Lorentz-invariant light-cone formulation)—this would just double the residue of the pole. Nor can closed strings be thought of as the usual sort of bound states (e.g., the hydrogen atom), since they appear in perturbation theory. Where are the external closed strings in Witten's theory?

Some light was shed on this issue by Horowitz *et al.*[3] Among other things, we constructed explicitly the open-string field which shifts between classical solutions of the string equation of motion—including those with differing metrics. It should then be the case that, as in any classical field theory, the infinitesimal shifts correspond to linearized particle excitations.[4] This leads to explicit expressions for *both* open- and closed-string states. The closed-string states, however, are *not* in the open-string Fock space. Thus, although they are perturbative in the string loop coupling constant, they are "nonperturbative" in the first-quantized level number. This implies that their couplings are, in principle, subject to associativity anomalies[5] and their generalizations.[6] In fact these anomalies turn out to be very important in this context. One consequence is that the cubic action and Witten's action are not equivalent for closed-string states. These states *do* obey the linearized cubic equation (at least level by level), but they do not obey the linearized form of Witten's equation ($QA=0$). In this paper, I shall describe these states, and demonstrate that they obey (level by level) the linearized equation of motion of the cubic action and that they couple correctly to open strings. I conclude with a discussion of several issues concerning closed strings that remain unresolved by our investigations.

I first review the relevant results of Ref. 3. In order to highlight the main idea, I temporarily sidestep the important issues of contour orderings and midpoint anomalies by restricting the following paragraph to fields in the open-string Fock space. A more careful discussion appears elsewhere.[6] The cubic action constructed from the string field A is then

$$S=\tfrac{1}{3}\int A*A*A. \tag{1}$$

The classical equation of motion following from (1) is

$$\delta S/\delta A=A*A=0.$$

For every conformally invariant two-dimensional string sigma model, a solution of this equation can be constructed as follows. Let j_α^B be the Becchi-Rouet-Stora-Tyutin (BRST) current associated with the sigma model. α is a world-sheet index and the superscript B is appended to remind the reader of the explicit dependence on the background fields B of the sigma-model target space. B corresponds to a solution of the classical string equation if the sigma model is conformally invariant. Now define

$$Q_L^B=\int_0^{\pi/2} j_\alpha^B(\sigma)\varepsilon^\alpha{}_\beta d\sigma^\beta$$

as the integral of the BRST charge density over the left half of the string. Q_R^B may be similarly defined. Let I denote the identity element of the star algebra ($I*A=A*I=A\ \forall A$). Then the string field $(Q_LI)^B$ is a solution of the classical equation $A*A=0$. This follows from the identities

$$Q_R^B A * A' + (-)^A A * Q_L^B A' = 0, \quad \{Q^B, Q_R^B\} = 0,$$

for sufficiently well-behaved fields A, A'. These further imply (along with $QI = 0$)

$$(Q_L I)^B * F + F * (Q_L I)^B = Q^B F, \tag{2}$$

where F here and hereafter denotes a field in the open-string Fock space. To understand the significance of the solution $(Q_L I)^B$, expand the cubic action in terms of the shifted field $F = A - (Q_L I)^B$. The cubic action then takes the Witten form

$$S = \int \left(\tfrac{1}{2} F * Q^B F + \tfrac{1}{3} F * F * F \right).$$

(Later we shall see that this is not in general valid for fields outside the Fock space.) The open-string diagrams thus, according to arguments in Ref. 2, give the correct dual amplitudes in the background B. In short, the solution $(Q_L I)^B$ is the classical solution B represented as a string field.

We wish to obtain the expression for linearized particle excitations around a given background. As in all classical field theories, this is simply the linearized variation (in the appropriate direction) of the exact solution. For example, the flat-space graviton is obtained by linearizing of the flat-space solution with respect to a perturbation $\delta_h g_{\mu\nu} = E_{\mu\nu} e^{ik\cdot X}$, where $E_{\mu\nu}$ is a constant, symmetric, graviton polarization tensor obeying $\eta^{\mu\nu} E_{\mu\nu} = k^\mu E_{\mu\nu} = 0$ and $k^2 = 0$. Let us denote this state by $h \equiv \delta_h (Q_L I)$. The coupling of h to two on-shell open-string states, F, F' (obeying $QF = 0 = QF'$), is easily obtained by linearizing of the integration-by-parts identity (2). It is given by

$$g(F, F', h) = \int (F * F' * h + F * h * F') = \int F * (\delta_h Q) F'.$$

This is manifestly the correct expression since it is the linearized variation of the free open-string action $\frac{1}{2} \int F * Q^B F$ with respect to a change in the gravitational background. Later on, we shall see that it represents an integral of the graviton vertex operator across the open-string world sheet.

To proceed further, we need an explicit expression for h. This requires regulation of $(Q_L I)^B$ and then variation of the background. We use the techniques developed in Ref. 6. As shown in Fig. 1, the regulated state $(Q_L I)^B [X(\sigma), c(\sigma)]$ is defined by the path integral over a strip of width δ with boundary conditions $X(\sigma), c(\sigma)$ along the vertical sides and open-string boundary conditions on the bottom. There is an insertion of a contour integral of j^B which is terminated a distance ε from the curvature singularity at the midpoint M. The path integral is weighted by the action S^B. $(Q_L I)^B$ is the limit $\varepsilon, \delta \to 0$ of this state, which was shown in Ref. 6 to be well defined. Linearizing with respect to δ_h we obtain two terms: The first is simply the naive variation of the BRST current acting on I,

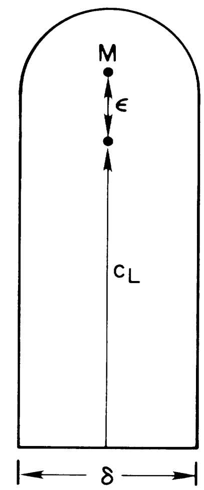

FIG. 1. The regulated state $(Q_L I)^B [X(\sigma), c(\sigma)]$ is defined by the functional integral weighted by the action S^B over the world-sheet shown. Open-string boundary conditions are applied along the horizontal boundary of length δ. $(Q_L I)^B$ is then a functional of the remaining boundary conditions. The extrinsic curvature of both boundary segments, as computed with the world-sheet metric, vanishes. The world-sheet scalar curvature vanishes everywhere except at M, where there is a deficit angle of π. The BRST current j^B is integrated along the contour C_L.

$$\int_0^{\pi/2} d\sigma^\gamma \varepsilon^\alpha{}_\gamma c^\beta (\partial_\alpha X^\mu \partial_\beta X^\nu - \tfrac{1}{2} g_{\alpha\beta} \partial_\gamma X^\mu \partial^\gamma X^\nu) E_{\mu\nu} e^{ik\cdot X} I.$$

The second comes from varying S^B. This term naively vanishes as the strip width $\delta \to 0$ because $\delta_h S$ is integrated over the area of the strip. However, $\delta_h S$ and j have a singular operator product, and finite pieces do remain as $\delta \to 0$. A detailed calculation shows that the limit $\delta, \varepsilon \to 0$ is well defined and the final result is[7]

$$h \equiv \delta_h (Q_L I) = \left(\int_0^{\pi/2} d\sigma^\alpha \varepsilon_{\alpha\beta} c^\beta V_h \right) I,$$

where $V_h = \frac{1}{2} \partial_\alpha X^\mu \partial^\alpha X^\nu E_{\mu\nu} e^{ik\cdot X}$ is the standard graviton vertex operator. This is an explicit expression for the graviton in terms of open-string operators acting on the open-string vacuum.

We now wish to verify that h obeys the linearized equation of motion of the cubic action. This equation is somewhat subtle because it involves contour integrals of operators which terminate at the string midpoint. Singularities may occur and the star product may depend on the order in which the contours approach the midpoint. In Ref. 6 I defined for this reason a graded commutator for states of the form $A = \int_0^{\pi/2} d\sigma^\alpha A_\alpha(\sigma) I$, $B = \int_0^{\pi/2} d\sigma^\alpha B_\alpha(\sigma) I$, where A_α and B_α are conformal fields, by

$$[A,B]=\lim_{\varepsilon\to 0^+}\left[\int_0^{\pi/2}d\sigma\cdot A(\sigma)I*\int_0^{\pi/2-\varepsilon}d\sigma'\cdot B(\sigma')I-(-)^{AB}\int_0^{\pi/2-\varepsilon}d\sigma'\cdot B(\sigma')I*\int_0^{\pi/2}d\sigma\cdot A(\sigma)I\right],$$

where $(-)^{AB}$ is -1 for A and B Grassmann odd operators and is $+1$ otherwise. If a symmetric contour ordering is chosen for the cubic action, the equation of motion is then *level by level*[6]

$$[A,A]=0. \tag{3}$$

What is meant by "level by level" is that, given a solution A_0 of this equation, the cubic action is stationary with respect to variations of the string which are in the Fock-space expansion around A_0. This is good enough if one only wishes to study open-string perturbations around the background A_0. However, one should also consider variations *outside* the Fock space since the perturbations corresponding to closed strings are not in the Fock space. This is not a simple matter. The correct class of variations must be determined, and there are subtleties in varying within a nonassociative product. I have not solved these problems. Equation (3) is therefore a necessary, but possibly not sufficient, condition for the full equation of motion to be satisfied. There could be an additional piece of the equation of motion which is "nonperturbative" in level number. Such a term might arise directly from our varying S, or from corrections to S involving only closed strings. In any case, $(Q_LI)^B$ is a solution level by level, i.e., $[(Q_LI)^B,(Q_LI)^B]=0$. The linearized form of this equation is

$$[Q_LI,h]+[h,Q_LI]=0.$$

The first term can be evaluated by completion of the BRST contour:

$$[Q_LI,h]=Qh.$$

By use of the relation for physical vertex operators $\{Q,\varepsilon_{\alpha\beta}c^\beta V_h\}=\partial_\alpha(c^\gamma c^\delta\varepsilon_{\gamma\delta}V_h)$, Qh may be expressed as

$$Qh=[c^\gamma c^\delta\varepsilon_{\gamma\delta}V_h]_0^{\pi/2}I.$$

The $\sigma=0$ piece vanishes by open-string boundary conditions. The midpoint contribution does not vanish. As can be seen from formulas in Ref. 5, the ghosts (acting on I) both diverge linearly as $\sigma\to\pi/2$, but V_h vanishes quadratically and Qh is therefore a finite, well-defined state. Thus Qh does not vanish because of midpoint anomalies. Qh is, however, in the center of the star algebra (that is, it commutes with sufficiently well-behaved states) because it is an operator acting on the identity at the midpoint. As will be seen momentarily, this is sufficient to insure gauge invariance of S-matrix elements.

The other piece of the equation of motion is $[h,Q_LI]=(\delta_hQ)Q_LI$, where (δ_hQ) is the graviton vertex operator integrated across the string. It can be shown that $[h,Q_LI]=-Qh$ and the level-by-level equation of motion is indeed satisfied.

In sum, the state $h=\delta_h(Q_LI)$ obeys level by level the linearized equation of motion of the cubic action but is not annihilated by Q. This implies that Witten's action and the cubic action are *not* equivalent for closed-string states.

Let us now return to the issue of BRST invariance of S-matrix elements, which appears in peril as $Qh\neq 0$. In computing open-closed couplings such as $g(F,F',h)$, we never actually need to know h. We only need the inner derivation generated by h, namely $[h,F]=(\delta_hQ)F$. Despite the fact that $Qh\neq 0$, this inner derivation does anticommute with Q on the Fock space:

$$[h,QF]=-Q[h,F].$$

This is the essential reason that second-quantized BRST Ward identities will not be violated by couplings to h. It is thus perhaps useful to think of closed strings as fields that generate inner derivations that commute with Q. From this perspective it is easy to see what goes wrong if we attempt to attach Chan-Paton factors[8] to h; the midpoint terms in $[h,F]$ will not in general cancel and h will not generate a BRST invariant inner derivation.

I wish to mention a circle of issues concerning closed strings that have not been resolved in this paper. Only the graviton has been explicitly constructed. Most other closed-string states can be similarly constructed, but qualitative differences may appear for the dilaton. $\delta I\neq 0$ for the dilaton because the world-sheet curvature singularity leads to a nonzero contribution from the $\int R^2\Phi$ dilaton coupling. This dependence of I on the dilaton background affects the coupling constant dependence of closed-string couplings, and may have further ramifications.

The closed-closed-open couplings can probably be computed in a similar manner. The closed-closed-closed tree-level coupling is more problematic. Expressions of the form $\int h*h'*h''$ correspond, at least naively, to functional integrals with one open-string boundary, which is not the tree-level coupling. Perhaps one must add a purely closed-string term to the action to obtain a fully consistent theory.

A perhaps not unrelated issue is Chan-Paton factors. Although a closed-string state with Chan-Paton factors does not generate a BRST invariant inner derivation, it does obey the level-by-level equation of motion $[Q_LI,h^A]+[h^A,Q_LI]=0$ where the Chan-Paton factor A just goes along for the ride. It is important to understand from first principles how these states are excluded.

In closing, there is a simple qualitative way of understanding why closed strings might appear as "nonperturbative" states outside of the open-string Fock space. An open-string configuration may be expanded in terms of oscillators as $X(\sigma)=\sum X_n \cos n\sigma$, on the interval 0 to π. Cosines (not sines) are used because of the open-string boundary conditions $\partial X(\sigma)/\partial\sigma=0$ at $\sigma=0,\pi$. However, the cosines are a complete set of functions on the open interval 0 to π. This means that, for any smooth function $f(\sigma)$ on $0<\sigma<\pi$, we can find coefficients $X_n[f]$ such that the cosine series converges pointwise (but not necessarily uniformly) to $f(\sigma)$. In particular, there are series that converge pointwise to closed-string configurations. These necessarily have an infinite number of nonzero X_n's, i.e., they are out of the open-string Fock space.

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[1]E. Witten, Nucl. Phys. **276**, 291 (1986); S. Giddings, E. Martinec and E. Witten, Phys. Lett. **176B**, 362 (1986).

[2]E. Witten, Nucl. Phys. **268**, 253 (1986).

[3]G. Horowitz, J. Lykken, R. Rohm, and A. Strominger, Phys. Rev. Lett. **57**, 283 (1986).

[4]This point has been stressed by T. Banks and E. Martinec.

[5]G. Horowitz and A. Strominger, Phys. Lett. B (to be published).

[6]A. Strominger, Phys. Lett. B (to be published).

[7]A simpler derivation of this result has been found by G. Horowitz and R. Woodard, unpublished.

[8]J. E. Paton and Chan H.-M., Nucl. Phys. **B10**, 516 (1969).