Nonlinear Plasma Dynamics in the Plasma Wake-Field Accelerator

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Excitation of nonlinear plasma oscillations by an intense ultrarelativistic electron beam is considered. It is shown, by analytical solutions of the one-dimensional relativistic fluid equations, that under certain conditions on the relative electron-beam and plasma densities extremely large longitudinal electric fields can be generated in the beam's wake. This scheme is a nonlinear version of the plasma wake-field accelerator, and is predicted to have advantages over the linear regime.

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Considerable interest has been stimulated in the last few years by the promise of extremely high accelerating gradients generated in plasma oscillations excited with a phase velocity near the speed of light by either lasers or bunched relativistic electron beams. The latter case, termed the plasma wake-field accelerator (PWFA), has been the subject of much theoretical investigation recently.[1-3] The majority of the analysis has been conducted under the assumptions necessary for linearization of the fluid equations. This analysis allows calculation of the longitudinal and transverse forces on both the driving and accelerating beams. The transformer ratio, the ratio of the maximum accelerating field behind the driving bunch to the maximum decelerating field inside the driving bunch, is shown to be less than or equal to 2 for longitudinally symmetric beam current profiles. This limitation can be circumvented by use of asymmetric profiles,[4] although some of the assumed profiles may be difficult to realize experimentally.

Fully relativistic nonlinear longitudinal plasma oscillations have been considered by Akhiezer and Polovin,[5] Noble,[6] and recently by Ruth *et al.*[3] and Amatuni, Elbakian, and Sekhpossian.[7] The behavior of the oscillations in the absence of the exciting electron bunch is considered in detail in these references. The behavior of these oscillations, for wave phase velocity $v_{\rm ph}=c$, can be qualitatively described as follows[6,8]: For small-amplitude waves the linearized solution is approximated; the deviation n_1 of electron density n from its equilibrium value n_0 is sinusoidal with plasma frequency $\omega_p =(4\pi e^2 n_0/m_e)^{1/2}$, as is the electric field. In the large-amplitude case, the electron density wave steepens with a large positive excursion in n_1 occurring for a short time during the oscillation followed by a much longer time during which n_1 approaches $-n_0/2$, and the electric field profile takes on a sawtooth appearance. The local oscillation frequency is smaller than the plasma frequency when n_1 is negative, mainly a result of the relativistic mass increase of the plasma electrons. When n_1 is positive, the oscillation frequency is larger than the plasma frequency because of the large local density of plasma electrons, an effect which opposes and overcomes the relativistic mass increase. The net oscillation period increases with amplitude.

The free nonlinear oscillations are well understood; the creation of such waves is the subject of this communication. We discuss herein the useful nonlinear attributes of large-amplitude electrostatic plasma waves excited in the wake of an intense ultrarelativistic electron beam.

The equations for nonlinear electron oscillations in a cold, collisionless plasma with stationary ions have been obtained previously in Refs. 5–8. If we include the effects of an electron beam of density n_b and velocity $\boldsymbol{\beta}_b c$ on the plama, the fluid equations containing the electron density, velocity $\mathbf{v}=\boldsymbol{\beta}c$, and the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are

$$\frac{\partial \mathbf{p}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{p}=-e(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}),\quad \mathbf{p}=\frac{m_e\boldsymbol{\beta}c}{(1-\beta^2)^{1/2}},$$
$$\nabla\cdot\mathbf{E}=4\pi e(n_0-n-n_b),\quad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\qquad (1)$$
$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{B}=-4\pi e(n\boldsymbol{\beta}+n_b\boldsymbol{\beta}_b)+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},$$

where $\mathbf{p}$ is the electron momentum.

Choosing the direction of propagation to be the z axis and assuming the wave motion is longitudinal and a function only of the variable $\tau=\omega_p(t-z/v_{\rm ph})$, we obtain the one-dimensional fluid equations:

$$n=\frac{n_0\beta_{\rm ph}}{\beta_{\rm ph}-\beta},\qquad (2)$$

$$\frac{d^2}{d\tau^2}\left[\frac{1-\beta_{\rm ph}\beta}{(1-\beta^2)^{1/2}}\right]=\beta_{\rm ph}^2\left[\frac{\beta}{\beta_{\rm ph}-\beta}+\frac{n_b}{n_0}\right].\qquad (3)$$

Note that the phase velocity of the excited plasma wave is determined by the velocity of the driving beam, $v_{\rm ph}=\beta_b c$. The electric field is purely longitudinal, $\mathbf{E}=E\hat{\mathbf{z}}$, and $\mathbf{B}=0$ for this driven electrostatic oscillation.

Since we are interested in these waves for use in a high-energy physics accelerator, we wish to study the fluid equations in the limit that $\beta_b\to 1$, i.e., the driving beam is ultrarelativistic. With a change of dependent variable

$$x(\tau) \equiv \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \quad (4)$$

and defining $\alpha = n_b/n_0$, Eq. (3) becomes

$$x''(\tau) = \frac{1}{2}\left(\frac{1}{x^2} - 1 + 2\alpha\right), \quad (5)$$

where the prime indicates differentiation with respect to τ.

We now consider this equation with an electron-beam bunch whose longitudinal density profile is flat over the full beam length l_b, i.e., α is constant for $0 \le ct - z \le l_b$ and zero elsewhere. There is no oscillation for $\tau < 0$, so $\beta = 0$ and the initial conditions are $x(0) = 1$, $x'(0) = 0$. The first integral of Eq. (5) is then

$$[x'(\tau)]^2 = 2(1-\alpha) - 1/x - (1-2\alpha)x. \quad (6)$$

Oscillatory solutions for x exist for $\alpha < \frac{1}{2}$, and the turning points of $x(\tau)$ occur at $x(\tau) = 1$, $1/(1-2\alpha)$. These points correspond to perturbed plasma electron densities of $n_1 = n - n_0 = 0$, $-2n_b(1-\alpha)$. We obtain immediately from the first of Eqs. (1) the electric field inside the beam as a function of $x(\tau)$

$$E(x) = -(m_e c\omega_p/e)(x') = \pm(mc\omega_p/e)[2(1-\alpha) - 1/x - (1-2\alpha)x]^{1/2}. \quad (7)$$

We now see that x is proportional to the electrostatic potential.

Integrating Eq. (6), we have

$$\tau(x) = (1-2\alpha)^{-1/2}\int_1^x \left(\frac{x}{(x-1)[(1-2\alpha)^{-1} - x]}\right)^{1/2} dx = 2(1-2\alpha)^{-1/2}E(\Psi,k), \quad (8)$$

where $E(\Psi,k)$ is the incomplete elliptic integral of the second kind and

$$\Psi = \sin^{-1}[(x-1)(1-2\alpha)]^{1/2}, \quad k^2 = 2\alpha. \quad (9)$$

The frequency of the driven oscillation is

$$\omega = [\pi(1-2\alpha)^{1/2}/2E(k)]\omega_p, \quad (10)$$

where $E(k)$ is the complete elliptic integral of the second kind and k is as defined above. The frequency is approximately ω_p for $\alpha \ll 1$ and decreases monotonically with increasing α, approaching zero as α nears $\frac{1}{2}$.

From Eqs. (6)–(10), we see that when the beam density approaches one-half the plasma density, the plasma oscillation period gets very large, and the electric field near the turning point at $n_1 = -n_b$ (very large x) approaches the linear wave-breaking limit, $E \to m_e c\omega_p/e \cong [n_0/(1\ \mathrm{cm}^{-3})]^{1/2}$ V/cm. This is the largest electric field obtainable inside any driving bunch of density $n_b \le n_0/2$, but is not, as will be shown subsequently, the maximum accelerating field behind the bunch.

To take the most physically interesting and mathematically transparent case we explore further the condition $n_b = n_0/2$. Then Eq. (6) simplifies and is integrated to find x as an implicit function of τ,

$$\tau = x^{1/2}(x-1)^{1/2} + \ln[(x-1)^{1/2} + x^{1/2}]. \quad (11)$$

In the limit of large x (in a long bunch), τ becomes approximately equal to x.

The continuity conditions on x and x' allow the calculation of the oscillation amplitude and electric fields in the wake of the bunch. For the homogeneous equation ($\alpha = 0$) there exists from Eq. (6) an invariant, $\gamma_m = \gamma + \frac{1}{2}(x')^2$, where $\gamma = (x + x^{-1})/2$ is the Lorentz factor of the plasma electrons and γ_m is its maximum value.

The electric field behind the beam is thus, evaluating γ_m at $\tau_f = 2\pi l_b/\lambda_p$, with the plasma wavelength defined as $\lambda_p = 2\pi c/\omega_p$,

$$E = \pm(m_e c\omega_p/e)[2\gamma_m - (x + 1/x)]^{1/2}. \quad (12)$$

The maximum accelerating field amplitude behind the bunch is now

$$E_+ = (m_e c\omega_p/e)(x_f - 1)^{1/2}, \quad (13)$$

where $x_f = x(\tau_f)$ and we have used $\gamma_m = (x_f + 1)/2$.

The maximum decelerating field inside the bunch is given by

$$E_- = (m_e c\omega_p/e)(1 - 1/x_f)^{1/2}. \quad (14)$$

Thus the transformer ratio is $R = E_+/E_- = x_f^{1/2}$. The transformer ratio approaches unity for a short beam and becomes approximately equal to $(2\pi l_b/\lambda_p)^{1/2}$ for $l_b \gg \lambda_p$.

A numerical example is plotted in Fig. 1, with transformer ratio $R = 4$ ($x_f = 16$). The beam charge is almost entirely neutralized by the excess plasma charge, so the electric field inside approaches an asymptote, as noted previously, $E_- \to m_e c\omega_p/e$. The plasma electrons continue to gain energy because of this nearly constant electric field, however, and the oscillation behind the bunch is driven to large amplitude.

The significance of the case $n_b = n_0/2$ is apparent from this example. High transformer ratios are obtained when the decelerating field inside the beam are as nearly constant as possible.[4] The perturbed plasma electron density has a lower limit from Eq. (2) of $-n_0/2$. If $n_b = n_0/2$ then the beam can be at best charge neutralized by the plasma electrons, and oscillatory behavior does not occur. If $n_b < n_0/2$ then the electric field inside the beam will oscillate; if $n_b > n_0/2$ the beam will not be completely neutralized and the electric field grows

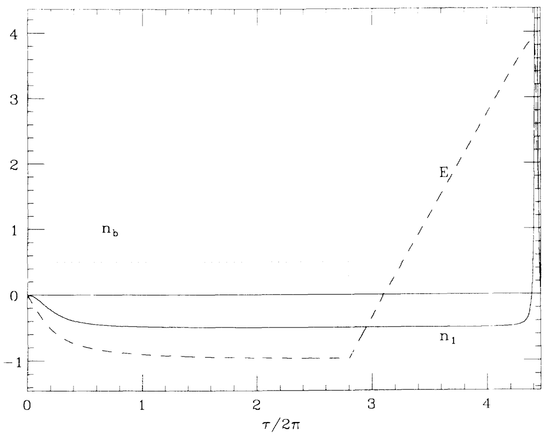

FIG. 1. Nonlinear plasma wave driven by beam of density $n_0/2$. Perturbed electron density, driving beam density, and electric field plotted as a function of $\tau/2\pi$. Densities are normalized to n_0; electric field is normalized to $m_e c\omega_p/e$. For this case $x_f=16$, $R=4$, and the maximum $n_1/n_0=143$.

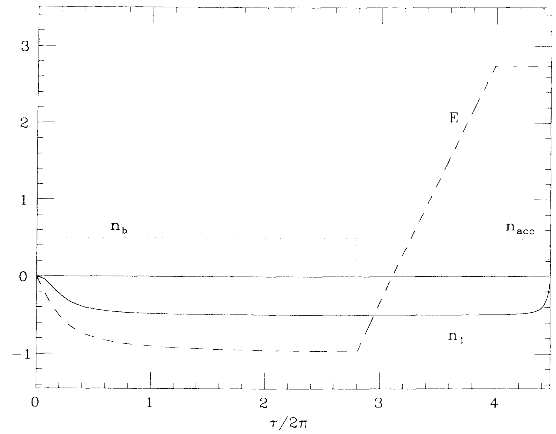

FIG. 2. Nonlinear plasma wave driven by beam of density $n_0/2$, with $x_f=16$, loaded at constant electric field by accelerating beam with efficiency $\eta=0.5$. Perturbed electron density, driving and accelerating beam density, and electric field plotted as a function of $\tau/2\pi$. Densities are normalized to n_0; electric field is normalized to $m_e c\omega_p/e$.

without approaching an asymptote. In either case, the variation of the decelerating field degrades the transformer ratio.

We now evaluate the effects of beam loading on the plasma dynamics and the implications that they hold for the inherent efficiency of the system. The plasma electron velocity should be negative at the beginning of the accelerating bunch, as otherwise the plasma electrons will work against the accelerating field, loading down the wave. This implies that x must be greater than unity, as can be seen by

$$n=\frac{n_0}{1-\beta}=\frac{n_0}{2}\left[1+\frac{1}{x^2}\right]. \qquad (15)$$

For the ideal case we take the electric field to be constant over the accelerating bunch. Thus for an accelerating bunch starting at τ_0 with $x(\tau_0)=x_0$, $x'(\tau_0)=x_0'$ =const, which requires $x''=0$. From Eq. (5), we now obtain an expression for the accelerating beam density $n_{\rm acc}$ as a function of $\Delta\tau=\tau-\tau_0$,

$$n_{\rm acc}=\frac{n_0}{2}\left[1-\frac{1}{x^2}\right]=\frac{n_0}{2}[1-(x_0+x_0'\Delta\tau)^{-2}]. \qquad (16)$$

From Eq. (12), we obtain $x_0'^2=R^2+1-x_0-1/x_0$, where R is the transformer ratio. The efficiency η of energy transfer from a single decelerating bunch to a single accelerating bunch can now be calculated,

$$\eta=\frac{x_0'\int_0^{x_0'/[R^2-(x_0')^2]}d(\Delta\tau)n_{\rm acc}(\Delta\tau)}{n_b\int_0^{\tau_f}d\tau\, x'(\tau)}=1-\varepsilon^2, \qquad (17)$$

where $\varepsilon=x_0'/x_{0,\rm max}'$, the fraction of the maximum available field used for acceleration. This is identical to the expression derived from the linear calculation.[2] It is less efficient to extract energy from the plasma wave at higher fields, as more energy remains in the wave after the passing of the accelerating bunch. An example of a driving bunch and accelerating bunch system is shown in Fig. 2, with the same driving bunch parameters as in Fig. 1.

There are some distinct advantages of using this regime in the PWFA over the alternatives presently being discussed. There is no singularity required in density profile to provide the desired transformer ratio, unlike the optimum linear case proposed by Bane, Chen, and Wilson.[4] The transformer ratio in our case scales approximately as the square root of the number of electrons N, in the driving beam; this is the same dependence as in the optimum linear scheme if the decelerating gradient in the driver is held constant. The accelerating gradients are inherently higher for a given plasma density, as one is no longer constrained to work far beneath the wave-breaking limit. This ameliorates the beam emittance blowup due to multiple scattering in the plasma and, along with the plasma nonlinearity,[6,8] increases the oscillation wavelength. Thus the accelerating bunch is not required to be as short. The effects that lead to high transformer ratios are not critically dependent on the condition $n_b=n_0/2$ holding exactly, and an acceptable density distribution should not be difficult to achieve experimentally.

Some foreseeable difficulties with this scheme, as with the PWFA in general, center on the three-dimensional effects, such as driver self-pinching, and transverse variation of accelerating fields. If the plasma is not driven by a beam with a flat radial profile of radius much greater than λ_p then the problem is no longer one dimensional. Our analysis predicts almost total charge and current neutralization of the driving beam after a plasma wavelength from the front of the beam. If this is also true in the three-dimensional case the potentially detrimental self-pinching forces of the driver can be diminished.

Problems not addressed in this treatment include ion motion and the effects of the finite electron temperature. A large-amplitude plasma wave such as we are considering can accelerate thermal electrons out of the plasma and trap them in its electrostatic potential well. These electrons load down the wave and provide a mechanism for saturation of the wave amplitude. For nonlinear plasma waves of the type described here any plasma electron with initial velocity component $\beta_0 c$ parallel to the wave vector will be trapped if[9] $\beta_0 \gtrsim (2\gamma_m)^{-1}$. In our numerical example calculations indicate trapping is insignificant if the plasma has an electron temperature $kT_e \ll 1$ keV. Further investigation of all these stated concerns is required, both theoretically and experimentally.

One might also anticipate that the generation of these large accelerating gradients requires a considerable amount of beam charge in the driver. If we require the beam radius to be equal to $2\lambda_p$, to keep radial effects small, an estimate on N for the numerical example in Fig. 1 is $N = 2\times 10^{13}$, where we have taken $\lambda_p = 1$ mm ($n_0 \simeq 10^{15}$ cm^{-3}). This is a prodigious amount of beam charge, considerably larger than current technology provides, but may be what is required to reach very high accelerating gradients, which in this case are predicted to be 12 GV/m.

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