Analysis of Subgrid-Scale Eddy Viscosity with Use of Results from Direct Numerical Simulations

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Without resort to any modeling, subgrid-scale eddy viscosity is computed from the results of high-resolution (64^3 and 128^3 grid points) direct numerical simulations of three-dimensional homogeneous, isotropic, decaying turbulence. In these simulations the eddy viscosity peaks sharply at the cutoff wave number, in rough agreement with the results of Kraichnan. In addition, in the low-wave-number range the eddy viscosity may be negative, contrary to the generally accepted concept of a subgrid-scale eddy viscosity. Some possible explanations of this behavior are discussed.

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As a result of the large number of excited modes in high-Reynolds-number turbulent flows and to the inherent limitations of even the fastest modern computers, it is possible to solve the Navier-Stokes equations numerically only for low- to moderate-Reynolds-number flows, $R_\lambda \lesssim O(10^2)$, where R_λ is the Reynolds number based on the Taylor microscale. For higher Reynolds numbers, subgrid-scale modeling is often used with the large-scale motions (i.e., for wave numbers $|\mathbf{k}| < k_c$, k_c being the cutoff wave number) computed explicitly from the Navier-Stokes equations and the effect of small scales (with wave numbers $|\mathbf{k}| > k_c$) approximated by a subgrid-scale eddy viscosity model.[1-3] The eddy viscosity models the process of the energy transfer from large ($|\mathbf{k}| < k_c$) to small ($|\mathbf{k}| > k_c$) scales by increased dissipation of the large scales. The purpose of this work is to investigate the concept of an eddy viscosity using results of direct numerical simulations of homogeneous isotropic turbulence.

The Navier-Stokes equations for the velocity field u_n in spectral form,

$$[(\partial/\partial t)+\nu k^2]u_n(\mathbf{k})=(-i/2)P_{new}(\mathbf{k})\int d^3p\, u_e(\mathbf{p},t)u_w(\mathbf{k}-\mathbf{p},t), \quad ik_n u_n=0, \tag{1}$$

lead to an equation for energy amplitudes,

$$\partial|u(\mathbf{k})|^2/\partial t=-2\nu k^2|u(\mathbf{k})|^2+\mathrm{Im}\left[u_n^*(\mathbf{k})P_{new}(\mathbf{k})\int d^3p\, u_e(\mathbf{p})u_w(\mathbf{k}-\mathbf{p})\right], \tag{2}$$

where

$$|u(\mathbf{k})|^2=u_n(\mathbf{k})u_n^*(\mathbf{k}), \quad P_{new}(\mathbf{k})=k_w(\delta_{ne}-k_nk_e/k^2)+k_e(\delta_{nw}-k_nk_w/k^2), \tag{3}$$

and the summation convention is assumed. The bracketed term represents nonlinear energy transfer and can be written as follows:

$$T(\mathbf{k})=[T(\mathbf{k})-T^-(\mathbf{k})]+T^-(\mathbf{k}), \tag{4}$$

where

$$T(\mathbf{k})=\mathrm{Im}\left[u_n^*P_{new}(\mathbf{k})\int d^3p\, u_e(\mathbf{p})u_w(\mathbf{k}-\mathbf{p})\right], \tag{5}$$

$$T^-(\mathbf{k})=\mathrm{Im}\left[u_n^*P_{new}(\mathbf{k})\int d^3p\, u_e(\mathbf{p})u_w(\mathbf{k}-\mathbf{p})\right], \quad |\mathbf{k}|,|\mathbf{p}|,|\mathbf{k}-\mathbf{p}|<k_c. \tag{6}$$

In Eq. (6) all wave numbers lie below the prescribed cutoff wave number k_c. Since $T^-(\mathbf{k})$ represents energy transfer from the mode $\mathbf{k}$ to all modes with wave numbers less than k_c, $T(\mathbf{k})-T^-(\mathbf{k})$ represents energy exchange between the mode $\mathbf{k}$ and two other modes such that at least one of them lies above the cutoff k_c. In an eddy-viscosity approach, one attempts to model this term by the following formula:

$$T(k|k_c)=-2\nu(k|k_c)k^2E(k), \quad k\le k_c, \tag{7}$$

where

$$T(k|k_c)\equiv 4\pi k^2\langle T(\mathbf{k})-T^-(\mathbf{k})\rangle, \tag{8a}$$

$$E(k)\equiv 4\pi k^2\tfrac{1}{2}\langle|u(\mathbf{k})|^2\rangle. \tag{8b}$$

$\langle\ldots\rangle$ denotes averaging over thin spherical shells of radius k, and $\nu(k|k_c)$ is the eddy viscosity. (In the restricted sense defined here, the term "eddy-viscosity modeling" used in this paper is also denoted as

"subgrid-scale modeling" in the literature.) In the isotropic average of Eq. (2), the term in Eq. (7) may be lumped together with the dissipation term giving a closed equation for the energy spectrum $E(k,t)$ determined on wave numbers $k \leq k_c$ once the form for $\nu(k \mid k_c)$ is determined. Similarly, in the Navier-Stokes equations (1), the effect of small scales may be modeled by the term $\nu(k \mid k_c)k^2 u_n(\mathbf{k})$, giving a closed equation for the large scales ($|\mathbf{k}| < k_c$).

The properties of the eddy viscosity $\nu(k \mid k_c)$ were investigated by Kraichnan[1] in the framework of the test field model and by Chollet and Lesieur[3] in the framework of the eddy damped quasinormal approximation. For an energy spectrum with an infinite inertial range, these theories predict a form for the normalized eddy viscosity, defined by

$$\nu_t^+(k \mid k_c) = \nu(k \mid k_c)/[E(k_c)^{1/2} k_c^{-1/2}], \quad (9)$$

which is included in Figs. 1 and 2. It is constant for $k \lesssim 0.4k_c$ and rises sharply in the vicinity of the cutoff k_c, indicating the importance of the local interactions.

In this paper, use is made of the following observation to compute the eddy viscosity without any modeling. If the accurately resolved velocity field $u_n(\mathbf{k},t)$ is known from direct numerical simulations, the eddy viscosity (9) may be evaluated for a particular truncation wave number k_c by employing formulas (5) and (6) and definitions (7) and (8).

Given the velocity field $u_n(\mathbf{k})$, the convolutions in formulas (5) and (6) are most effectively computed by use of a fast-Fourier-transform algorithm on the convolved fields, computing the local products in physical space, and transforming their product back to spectral space.[4] This method requires 3 orders of magnitude less work than direct summation for the 64^3 field (4 orders of magnitude for the 128^3 field). The numerical code was checked against exact relations obtained by assuming Gaussian forms for the velocity field, which allow analytical evaluation. Direct numerical simulations of homogeneous, isotropic, decaying turbulence were performed for different initial conditions on 64^3 and 128^3 meshes, and the eddy viscosity was evaluated from velocity fields at different times of evolution [corresponding to prescribed Reynolds numbers $R_\lambda(t)$] and with different wave-number truncations k_c. Runs F64E, F128D, and F128L (see Table I) are representative of the seven simulations that were performed and studied in detail.

Runs F64E and F128D were initialized with a random velocity field with an energy spectrum

$$E(k,0) = 16(2/\pi)^{1/2} v_0^2 k_p^{-5} k^4 \exp[-2(k/k_p)^2] \quad (10)$$

which is peaked at $k = k_p$. For the spectrum (10) the Taylor microscale is $\lambda = 2/k_p$, the internal Reynolds number is $R_\lambda = 2v_0/(\nu k_p)$, and v_0 is the rms turbulent velocity. Run F128L was started with an initial spectrum taken from the laboratory measurements of Comte-Bellot and Corrsin[5] (Table 2 in their paper; data at nondimensional time $tU_0/M = 385$). The unit of length was changed so that the lowest measured wave number k is always 1, with the corresponding rescaling of all physical quantities obtained by substituting cen-

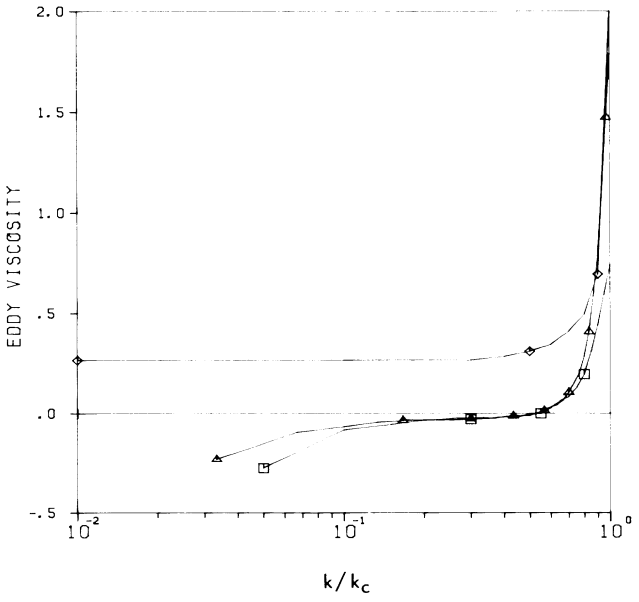

FIG. 1. Eddy viscosity computed from a direct numerical simulation of homogeneous, isotropic turbulence for case F128D4 (cf. Table I). Two different wave-number cutoffs, k_c, are shown: squares, $k_c = 20$; triangles, $k_c = 30$; lozenges, Kraichnan's theory.

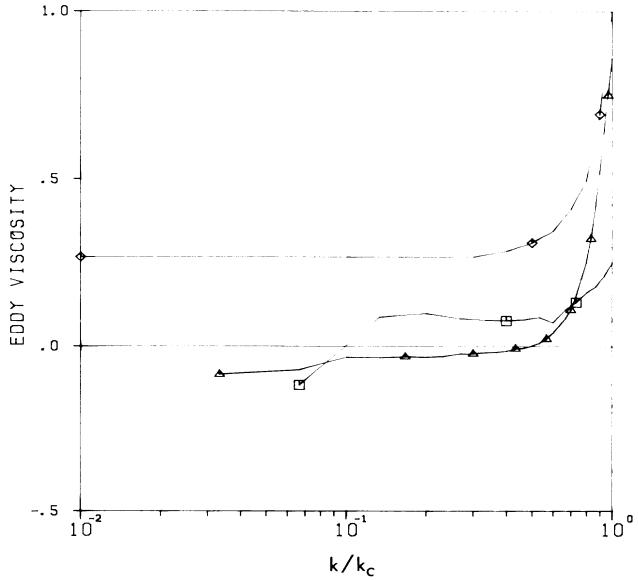

FIG. 2. Same as Fig. 1 but for case F128L5: squares, $k_c = 15$; triangles, $k_c = 30$; lozenges, Kraichnan's theory.

TABLE I. Relevant parameters for the simulations. (S is the velocity derivative skewness and N is the number of modes; the suffix "0" on the run number refers to the initial conditions.)

Run	R_λ	λ	v_0	k_{max}	N	ν	k_p	S	t
F64E0	84	0.84	1	32	64^3	0.01	2.38		
F64E1	46.5	0.53	0.87						
F64E2	36.6	0.47	0.78						
F64E3	27.6	0.44	0.63						
F128D0	84	0.42	2	64	128^3	0.01	4.76		0.0
F128D4	40.3	0.24	1.67					0.48	0.322
F128D6	29.7	0.22	1.34					0.48	0.554
F128L0	36	0.12	0.45	64	128^3	0.0015			0.0
F128L1	34.3	0.12	0.44					0.37	0.303
F128L2	29.8	0.12	0.37					0.36	0.607
F128L3	25.7	0.14	0.28					0.41	1.187
F128L4	23.0	0.16	0.22					0.45	1.921
F128L5	20.7	0.18	0.17					0.48	2.892

timeters for $\alpha[L]$, where $[L]$ is the new unit length and $1\text{ cm}=\alpha[L]$. The most notable difference between these two initial conditions is the rapid decrease of the spectrum (10) for large wave numbers k when compared with the experimental spectra, which usually can be fitted by less rapidly decreasing curves for large wave numbers.[6] Some of the main parameters for the numerical simulations are gathered in Table I.

The behavior of normalized eddy viscosities (9) for a typical case initialized with the spectrum (10) is shown in Fig. 1. A general feature of these computed eddy viscosities is their steep increase in the vicinity of the cutoff k_c. The viscosity term $\nu_t^+(k\,|\,k_c)$ grows an order of magnitude as k ranges from $0.7k_c$ to k_c. This effect is consistent qualitatively with predictions of Kraichnan's theory[1] for the subgrid-scale eddy viscosity and indicates the importance of the local interactions at the cutoff. The value of the unnormalized viscosity $\nu(k\,|\,k_c)$ is usually comparable to the molecular viscosity in this range of wave numbers and Reynolds numbers.

Another feature observed in most of our results is negative eddy viscosities for $k<0.5k_c$ with values generally an order of magnitude less than the molecular viscosity. Physically, negative eddy viscosity means net transfer of the energy from the wave numbers above the cutoff to the wave numbers below $0.5k_c$. This feature contradicts the physical picture leading to the eddy-viscosity concept. This concept is based on the notion of the effective dissipation at the large scales via the energy transfer to small scales, in much the same way as the molecular viscosity is responsible for dissipation of hydrodynamic motions that occur at scales much greater than the mean free path of molecular motions.

The eddy viscosities for the experimental spectrum (Fig. 2) show behavior similar in many respects to that described above. For low cutoff wave number k_c, however, they are no longer consistently negative at low wave numbers k. But even in this case, the computed values of the eddy viscosity are significantly below Kraichnan's prediction (see Fig. 2). The averaged transfer spectra $\langle T(\mathbf{k})\rangle$ and $\langle T^-(\mathbf{k})\rangle$ on full and truncated meshes are shown in Fig. 3.

There are a few possible explanations for discrepancies between our results and Kraichnan's theory of the subgrid-scale eddy viscosity. Kraichnan assumed an infinite inertial range in obtaining his theoretical predictions, and in our calculations the Reynolds numbers of the turbulence are too low to obtain the inertial range behavior. In fact, the expression for the eddy viscosity

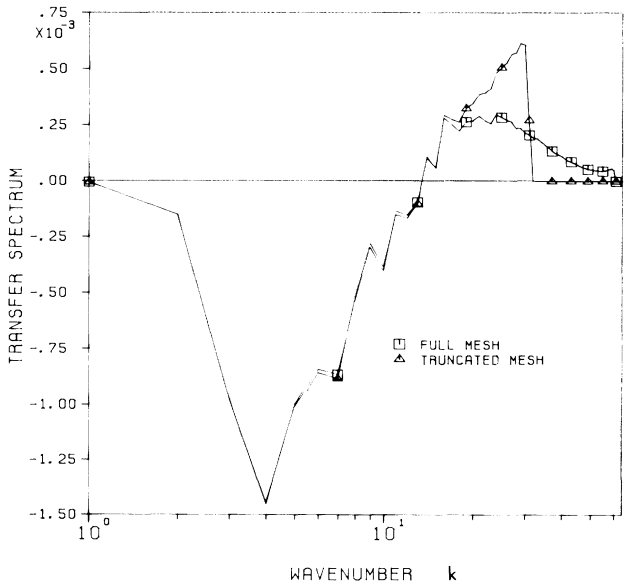

FIG. 3. The transfer spectrum for the case F128L5. The wave-number cutoff for the truncated mesh is $k_c=30$.

for $k \ll k_c$ derived by Kraichnan,[1]

$$\nu(k \mid k_c) = \tfrac{1}{15} \int_{k_c}^{\infty} \theta_{qqk} \left[5E(q) + q \frac{dE(q)}{dq} \right] dq \quad (11)$$

(θ_{qqk} is the characteristic time of triad interaction), indicates that it may be negative if the energy spectrum falls off faster than k^{-5} for k beyond k_c. We found that in our simulations the negative eddy viscosities were consistently associated with such behavior of the spectra for large wave numbers. Moreover we have only a limited wave-number range in our calculations, with the maximum wave number $k_{\max}$ not exceeding $64k_{\min}$ in 128^3 simulations, where $k_{\min}$ is the minimum wave number. Therefore, the separation of scales in our simulations may be too small to expect the asymptotic high-Reynolds-number results to be valid. Departures from the Kraichnan theory were also observed by Chollet and Lesieur[3] when the cutoff wave number k_c was very close to the peak in the energy spectrum (Fig. 3 in their paper).

The results of our work show that for low Reynolds numbers the eddy viscosity in homogeneous turbulence may be negative for $k \lesssim 0.5k_c$. The asymptotic analysis of the eddy viscosity using the inertial range spectra made by Kraichnan is not valid in this case. However, the eddy viscosity in our results for $k \simeq k_c$ behaves qualitatively in the same manner as in Kraichnan's analysis, indicating that the mechanism of the local transfer of energy in homogeneous turbulence is qualitatively the same at low as well as at high Reynolds numbers. Therefore, the analysis of the local energy transfer at low Reynolds numbers via direct numerical simulations may prove to be useful also for high-Reynolds-number flows. The discrepancy for $k \lesssim 0.5k_c$ between our results and Kraichnan's analysis, which assumes an infinite inertial range, indicates the possibility of errors when subgrid-scale models derived from the inertial range spectra are used for flows in which the Reynolds numbers or spatial resolution is too low to exhibit a significant inertial range. This is important since many laboratory and engineering flows have Reynolds numbers $R_\lambda \simeq 100$ to 200 and fall into this category. For geophysical flows with $R_\lambda = O(10^4)$, the existence of the inertial range spectrum is well documented and Kraichnan's asymptotic eddy viscosity may be applicable.

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