# Induced Gauge Fields in a Nongauged Quantum System

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It is shown that non-Abelian gauge fields arise in a nongauged quantum system in the adiabatic approximation, by the working out of a model of $N$-dimensional rotational symmetry. The induced gauge fields are symmetric under $N$-dimensional rotations accompanied by compensating gauge transformations of the group SO($N$).

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It was recently demonstrated[1-3] that gauge fields can appear very naturally in the adiabatic description of a quantum system, even if there is apparently no gauge symmetry in the Hamiltonian of the system. Consider a system described by the Hamiltonian $H(\boldsymbol{\lambda})$, depending on a set of continuous parameters $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots)$. The system varies adiabatically as parameters change from $\boldsymbol{\lambda}_i$ to $\boldsymbol{\lambda}_f$ during a time period from $t=0$ to $t=T$. Under the assumption $\boldsymbol{\lambda}_i=\boldsymbol{\lambda}_f$, the evolution of the system is given by the time-dependent Schrödinger's equation

$$i\,\partial\psi/\partial t=H(\boldsymbol{\lambda}(t))\psi, \tag{1}$$

with the boundary conditions $\boldsymbol{\lambda}(0)=\boldsymbol{\lambda}(T)$. Let $\psi_n(\lambda(t))$ be the stationary-state wave function

$$H\psi_n(\boldsymbol{\lambda}(t))=E_n(\boldsymbol{\lambda})\psi_n(\boldsymbol{\lambda}(t)), \tag{2}$$

and

$$H\psi_n(\boldsymbol{\lambda}(0))=E_n(\boldsymbol{\lambda}(0))\psi_n(\boldsymbol{\lambda}(0)). \tag{3}$$

The adiabatic theorem just tells us that if in a sufficiently long time period $\boldsymbol{\lambda}$ varies slowly from its initial value $\lambda_i$ to some other value, the system originally in a stationary state $E_n(\lambda(0))$ will remain there with the wave function changed up to a phase, provided the energy level does not cross other levels:

$$\psi(t)=\exp\left[i\int_0^t dt'E_n(\boldsymbol{\lambda}(t'))\right]e^{i\gamma_n}\psi_n(\boldsymbol{\lambda}(t)). \tag{4}$$

The phase factor $\exp(i\int dt'E_n)$ is dynamical, while there may be an additional phase $\lambda_n(t)$, satisfying

$$d\gamma_n(t)/dt=i\langle n(\boldsymbol{\lambda})\,|\,\nabla_{\boldsymbol{\lambda}}\,|\,n(\boldsymbol{\lambda})\rangle d\boldsymbol{\lambda}/dt. \tag{5}$$

We write $|n(\boldsymbol{\lambda})\rangle$ for $\psi_n(\boldsymbol{\lambda}(t))$. For $\boldsymbol{\lambda}(0)=\boldsymbol{\lambda}(T)$,

$$\gamma_n=i\oint_C\langle n(\boldsymbol{\lambda})\,|\,\nabla_{\boldsymbol{\lambda}}\,|\,n(\boldsymbol{\lambda})\rangle d\boldsymbol{\lambda}. \tag{6}$$

$C$ is a closed contour in the parameter space. It was generally accepted that $\gamma_n$ is an unimportant quantity and can be chosen arbitrarily by convention. However, as observed by Mead and Truhlar,[4] the determination of the Born-Oppenheimer nuclear three-body wave function presents some complications which require careful consideration of the phases of the wave function. But the general and deep meaning of this phase was not known until the analysis by Berry[1] in 1984. Berry pointed out that $\gamma_n$ is determined by the geometrical structure of the parameter space. It was immediately recognized by Simon[2] and by Niemi and Semenoff[3] that this phase is just the topological invariant of the Chern class, and is precisely the holonomy in a line bundle. Thus the topology in simple quantum-mechanical systems is uncovered. Wilczek and Zee,[5] generalizing the construction of Berry and Simon, showed that Abelian and non-Abelian gauge fields can arise in the adiabatic development of the quantum-mechanical system, though there was no gauge field or gauge symmetry present in the initial formulation of the system. Let us set

$$A_\mu=\langle n(\boldsymbol{\lambda})\,|\,\nabla_{\boldsymbol{\lambda}}\,|\,n(\boldsymbol{\lambda})\rangle;$$

then

$$\gamma_n=\oint_C A_\mu(\boldsymbol{\lambda})d\boldsymbol{\lambda}. \tag{7}$$

Examples of di-atom and spin precession were worked out.[6] Looking at the parameter space, we know that the Berry-Simon phase is related to the connection in the curved parameter space.

In this Letter, I analyze a model to show the induced non-Abelian gauge structure by working out explicitly the connections in parameter space. In this way, the geometric relation among the space of degenerate levels, the parameter space, and the configuration space can be demonstrated clearly.

Let us consider a model (Fig. 1). "Nucleus" $A$ is constrained to move on a spherical surface $S_A^{N-1}$ embedded in an $N$-dimensional configuration space. The points in this space are denoted by $\mathbf{x}(x_1,x_2,\ldots,x_N)$. A spinless "electron" $B$ interacting with the moving nucleus moves around $A$ on another spherical surface $S_B^{N-1}$ centered at $A$. The whole system respects SO($N$), $N$-dimensional rotational symmetry; the interactions between $A$ and $B$ are such that when $A$ is taken at a fixed position, $R$, the symmetry is reduced to SO($N-1$). The Hamiltonian is

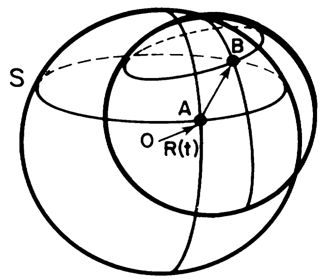


FIG. 1. Model of "rotating atom."

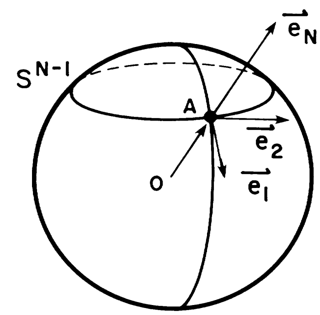


FIG. 2. Geometry of parameter space.

generally

$$H=(1/2M_A)\nabla_R^2+(1/2M_B)\nabla_r^2+V_A(R)+V_{AB}(\mathbf{R}-\gamma)+V_B(r), \quad (8)$$

where $\mathbf{R}$ and $\mathbf{r}$ are the position vectors of $A$ and $B$, respectively. The masses $M_A \gg M_B$ are assumed. We take $R(t)$ as the adiabatic parameter. The parameter space is then $S^{N-1}$ which in the present case coincides with $S_A^{N-1}$. This space of an $(N-1)$-dimensional spherical surface is just the coset space $SO(N)/SO(N-1)$. During an adiabatic evolution of the nucleus, $\mathbf{R}(t)$ traces a curve on $S_{N-1}$. We need to know the energy-level degeneracy of the stationary-state equation for an instantaneous $R$,

$$H(\mathbf{R})\,|n(\mathbf{R})\rangle=E_n\,|n(\mathbf{R})\rangle. \quad (9)$$

$H(\mathbf{R})$ denotes $H$ for an instantaneous $\mathbf{R}$ with the nucleus kinetic-energy term ignored. $|n(\mathbf{R})\rangle$ is the eigenfunction belonging to energy level $E_n$; it can also be written as $\phi_n(\mathbf{R},\mathbf{r})$. The total wave function of the whole system according to the adiabatic approximation is

$$\psi(\mathbf{R},\mathbf{r})=\chi_n(\mathbf{R})\phi_n(\mathbf{R},\mathbf{r}). \quad (10)$$

The eigenequation (9) describes the motion of the electron on an $(N-1)$-dimensional spherical surface $S_B^{N-1}$. The solutions of this type of equation are known to be the hyperspherical harmonics.[7] The eigenvalues are characterized by integer quantum numbers $m$; for spherical harmonics on an $N$-dimensional surface

$$E_{N,m}=\kappa m(N+m-1), \quad (11)$$

where $m$ is the degree of the spherical harmonic, and $\kappa$ is a constant which depends upon the numerical parameters in the Hamiltonian. The degeneracy for an $m$ level is[6]

$$d(N,m)=(2m+N-1)\frac{(m+N-2)!}{(N-1)!m!}. \quad (12)$$

So, for $m=0$, $d=1$ and the level is nondegenerate. For $m=1$, $d=N-1$ and the level is $N$-fold degenerate. In our model, therefore, there exists an $N$-fold degenerate level. This degeneracy exists for all $R(t)$ on the parameter space. Then we can associate a $N$-dimensional frame with each point in the parameter space, i.e., at every point on $S_A^{N-1}$ associate an $N$-dimensional coordinate system (Fig. 2). So far, there is no gauge symmetry imposed and there appears no gauge field in the formalism. As $\mathbf{R}(t)$ traces a closed contour in the parameter space $S^{N-1}$, the resulting extra phase in the wave function $\phi_n(\mathbf{R},\mathbf{r})$, the Berry-Simon phase, is related to connections of parallel displacement of the frame vectors on $S^{N-1}$. In fact, $\gamma_n$ is the integral of the connection along a closed contour on $S^{N-1}$ which measures the angular displacement resulting from parallel displacement. These connections define a non-Abelian gauge field $A_\mu^a$, $\mu=1,\ldots,N$, $a=1,\ldots,N$.

We note here that the Hamiltonian of our model was so constructed that the eigenfunctions of $H$, $\psi_n(\lambda)$, are complex functions. So $\gamma$ will be nonzero.

Let us introduce stereographic projections.[8] Points on the spherical surface are projected onto a tangent plane $\Pi_\eta$ at the north pole of the sphere. This is shown in Fig. 3; point $p$ on the unit sphere is projected to point $Q$ on $\Pi_\eta$.

Let $\eta_\mu$ be the stereographic projection coordinates, $\mu=1,\ldots,N-1$. These are the stereographic projections of a point $p$ on the unit spherical surface $S^{N-1}$ em-

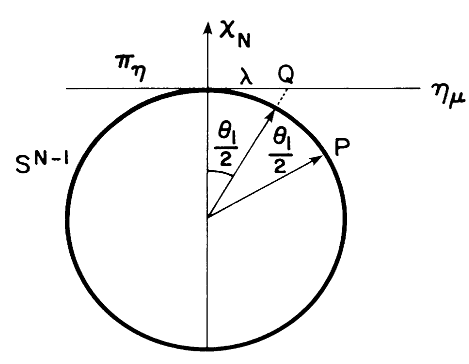


FIG. 3. Stereographic projection coordinates.

bedded in $N$-dimensional Euclidean space $E_N$ (Fig. 1):

$$\eta_1=\lambda\sin\theta_2\sin\theta_3\cdots\cos\theta_{N-1},$$
$$\eta_2=\lambda\sin\theta_2\sin\theta_3\cdots\sin\theta_{N-1},\qquad(13)$$
$$\vdots$$
$$\eta_{N-1}=\lambda\cos\theta_2,$$

where

$$\lambda=R\tan\theta_1/2$$

and $\theta_1,\ldots,\theta_{N-1}$ are the angular coordinates in the spherical coordinate system, and

$$P=(\theta_1,\theta_2,\ldots,\theta_{N-1}),$$
$$Q=(\eta_1,\eta_2,\ldots,\eta_{N-1}).$$

The projective coordinates are related to Cartesian coordinates, $P(x_1,\ldots,x_N)$ by

$$x_\mu=2\eta_\mu R/(1+\lambda^2),\quad \mu=1,\ldots,N-1,$$
$$x_N=R(1-\lambda^2)/(1+\lambda^2),\qquad(14)$$
$$R^2=x^a x_a,\quad a=1,\ldots,N.$$

$\mu,\nu$ are the projective coordinate indices and $a,b$ are Cartesian coordinate indices.

Using the stereographic projection coordinates, we express the metric tensor $g_{\mu\nu}(\eta)$ on $S^{N-1}$:

$$g_{\mu\nu}(\eta)=[4R^2/(1+\lambda^2)^2]\delta_{\mu\nu}.\qquad(15)$$

The natural connection $\{{}^{\rho}_{\mu\nu}\}$ of the parallel displacement can be evaluated from $g_{\mu\nu}$:

$$\begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}=\frac{1}{2}g^{\rho\tau}\left(\frac{\partial g_{\tau\nu}}{\partial\theta^\mu}+\frac{\partial g_{\tau\mu}}{\partial\theta^\nu}-\frac{\partial g_{\mu\nu}}{\partial\theta^\tau}\right).\qquad(16)$$

In terms of $\eta_{\mu\nu}$ we have

$$\{{}^{\rho}_{\mu\nu}\}=-[2/(1+\lambda^2)](\eta_\mu\delta_{\rho\nu}+\eta_\nu\delta_{\rho\mu}-\eta_\rho\delta_{\mu\nu}).\qquad(17)$$

These components of natural connection are then transformed into components referring to an orthonormal coordinate system. The results are

$$\Gamma^A_{\mu B}=[2/(1+\lambda^2)](\eta_A\delta_{\mu B}-\eta_B\delta_{\mu A})$$
$$=[2/(1+\lambda^2)]\eta_\mu(X_{\mu\nu})^A_B,\qquad(18)$$

where

$$(X_{\mu\nu})^A_B=\delta_{\mu B}\delta_{\nu A}-\delta_{\mu A}\delta_{\nu B},\quad \mu,\nu,A,B=1,\ldots,N-1;$$

$X_{\mu\nu}$ are the generators of SO$(N-1)$.

We have been working in the projective coordinates. We now transform back to Cartesian coordinates $x_a$ in Euclidean space $E^N$. Let us extend $\eta_\mu$, $\mu=1,\ldots,N-1$, to take one more coordinate $\eta_N$ with

$$\eta_N\equiv R.$$

From now on $\mu$ takes on values $1,2,\ldots,N$. We have

$$\Gamma^a_{cb}(x)=R^{-2}(x_a\delta_{bc}-x_b\delta_{ac})$$
$$=R^{-2}(x_a X_{cd})^a_b,\qquad(19)$$

where $X_{cd}$ are generators of the group SO$(N)$.

The gauge fields are defined by

$$\Gamma_a\equiv gW_a=\tfrac{1}{2}gW^{bc}_a X_{bc},$$

since

$$(X_{bc})_{ad}=\delta_{bd}\delta_{ca}-\delta_{ba}\delta_{cd},\qquad(20)$$

so that

$$W^{bc}_a(x)=-(1/gR^2)(x_b\delta_{ac}-x_c\delta_{ab}).\qquad(21)$$

Returning to Eq. (5), we have shown that non-Abelian gauge fields arise in the adiabatic process in the present model. It is clear that $W^{bc}_a(x)$ are symmetric with respect to rotation in configuration space and a compensating rotation in "iso-spin" space, i.e., a gauge transformation.

I further remark that this combined invariance property of gauge fields suggests that the conserved total angular momentum of the whole system consists of two parts: The orbital angular momentum in the ordinary sense and the "isospin" part,

$$\mathbf{J}=\mathbf{L}+\mathbf{I}.\qquad(22)$$

The results in the present paper may suggest that we can construct a model in which spin comes out from isospin, a phenomenon which was shown by Jackiw and Rebbi[9] and by Hasenfratz and 't Hooft[10] many years ago. However, in their investigations, a given monopole field is necessary, while the present model suggests that we can start with a nongauged quantum system under adiabatic evolution and an induced monopole field will arise. This should be interesting and further investigation will be pursued in another publication.[11]

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