String Theory as the Kähler Geometry of Loop Space

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We argue that the Kähler geometry of the loops on space-time describes bosonic string theory. The Kähler potential is the dynamical (field) variable of closed-bosonic-string theory. The equation of motion for this field is that a generalized Ricci tensor vanishes. Loops on flat space constitute a solution only in 26 dimensions.

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String theory[1] so far lacks a good geometric and nonperturbative formulation analogous to the Riemannian geometry of space-time in general relativity. String field theory is a natural approach to a nonperturbative formalism.[2-5]

An open string has a massless spin-1 excitation. Open strings, therefore, seem to be a source of matter fields. Closed strings, on the other hand, have a massless spin-2 excitation, and thus seem to be associated with gravity. In general relativity, gravity is described by the Riemannian geometry of space-time. In string theory, space-time is replaced by the space of loops in space-time. The space-time manifold itself is recovered by consideration of the constant loops—this is the long-wavelength limit of string theories in which we expect to recover conventional field theories. It is natural to expect then that the open-string field describes the propagation of matter and the closed string describes the geometry of loop space. The loop space[6,7] of real space-time is a complex differentiable manifold. The complex analogs of Riemannian manifolds are those in which vectors are transformed by the unitary group upon parallel transport (this preserves both the metric and the complex structure). These are the Kähler manifolds.[8]

The fundamental object of Kähler geometry is a real scalar field K, the Kähler potential. The dynamical variable of closed-string theory must then be a real scalar function in loop space. Geometrical considerations will lead us to a nonlinear equation of motion for this field which is the analog of Einstein's equations. This equation of motion is given by the vanishing of a certain generalization of the Ricci tensor. Flat loop space is a solution of this equation of motion only if the space-time dimension is 26, so that we recover known results at zeroth order. Our work gives, however, a nonperturbative equation of motion for closed-string theory, and may have solutions very different from 26-dimensional flat space. The nonlinearity of the equation naturally introduces interactions.

We introduce now the ingredients of our approach to string theory.

Let $\mathcal{L}R^{d-1,1}$ be the space of maps of the circle into Minkowski space $R^{d-1,1}$. The elements of $\mathcal{L}R^{d-1,1}$ are functions x on the interval $[-\pi,\pi]$ satisfying $x(-\pi)=x(\pi)$. We will also consider $\Omega R^{d-1,1}$, the subspace of $\mathcal{L}R^{d-1,1}$ given by loops beginning and ending at a fixed point, the origin. $\mathcal{L}R^{d-1,1}$ may be interpreted in two ways. It is the configuration space for closed strings and it is the phase space for open strings. The identification[1]

$$x(\sigma)=\begin{cases}(\dot{y}+y')(\sigma), & 0\le\sigma\le\pi,\\ (\dot{y}+y')(-\sigma), & -\pi\le\sigma\le 0,\end{cases}$$

where $y=[0,\pi]\to R^{d-1,1}$ is the open-string coordinate, makes this explicit.

$\Omega R^{d-1,1}$ has the structure of a complex manifold. To see this, make a Fourier expansion of $x\in\Omega R^{d-1,1}$,

$$x(\sigma)=\sum_{n\neq 0}x_n e^{in\sigma}-\sum_{n\neq 0}x_n e^{in\pi} \tag{1}$$

with $\bar{x}_n=x_{-n}$. The complex structure[6] J is defined by

$$(Jx)(\sigma)=-i\sum_{n\neq 0}\operatorname{sgn}(n)(x_n e^{in\sigma}-x_n e^{in\pi}). \tag{2}$$

$\mathcal{L}R^{d-1,1}$ is actually a family of complex manifolds labeled by the zero mode x_0: $\mathcal{L}R^{d-1,1}=\Omega R^{d-1,1}\times R^{d-1,1}$. It is at this point that we first see the appearance of Kähler geometry. $\Omega R^{d-1,1}$ is a Kähler manifold, with the Kähler form

$$\omega(u,v)=\frac{1}{2\pi}\int u^\mu(\sigma)v'^\nu(\sigma)\eta_{\mu\nu}d\sigma, \tag{3}$$

with the flat Minkowski metric $\eta^{\mu\nu}$. The corresponding Kähler potential is

$$K_0=\sum_{n=1}^{\infty}n\,|x_n|^2. \tag{4}$$

The wave function of the open string is an analytic function on $\mathcal{L}R^{d-1,1}$ (i.e., it is independent of x_n for $n>0$). The wave function for the closed string Φ is an arbitrary real function on $\mathcal{L}R^{d-1,1}$. We want to think of this as a perturbation of the "flat" Kähler potential K_0. It will, of course, only be the sum $K=K_0+\Phi$ which has true geometrical significance.

The strings we have considered so far (elements of

$\mathcal{L}R^{d-1,1}$) have a specific parametrization. Physical string amplitudes should be independent of this parametrization—in other words, invariant under the group of diffeomorphisms of the circle DiffS^1. Only the complex structure provided by the map J is affected by actions of DiffS^1. It is easy to see that pure rotations (S^1) do not change the complex structure. The space of all complex structures of $\mathcal{L}R^{d-1,1}$ transformed into each other by DiffS^{-1} forms a manifold $\mathcal{M}=\text{Diff}S^1/S^1$. This manifold thus plays a crucial role in string theory.

In string theory it is well known that the ghosts which arise in covariant quantization are an essential element.[3] We should consider then ghost excitations, in addition to the bosonic excitations. To treat these ghosts correctly one must specify a vacuum state—the filled Dirac sea. We digress a little to outline this point. Let V be the one-particle Hilbert space of a fermionic system, with positive-energy states (V^+) and negative-energy states (V^-), i.e., $V=V^+\oplus V^-$. The vacuum state is given by the (infinite) wedge product of all the basis elements of V^-. In other words, the vacuum is a one-dimensional vector space given by the densities of weight one in V^-, which we denote by $\bar{\Gamma}$. For the ghosts of string theory, the one-particle Hilbert space can be identified with the Lie algebra of DiffS^1, since they transform under the adjoint representation. $\bar{\Gamma}$ has been studied in this context in the mathematical literature.[9,10] Frenkel, Garland, and Zuckerman[9] have given a rigorous account of Becchi-Rouet-Stora-Tyutin (BRST) (covariant) quantization[3] by studying the associated cohomology. In their work, motivated by that of Banks and Peskin,[5] a puzzling similarity to Kähler geometry was noted. It can be shown[11] that the Kähler manifold $\mathcal{M}=\text{Diff}S^1/S^1$ has an invariant cohomology which is the BRST cohomology. We turn now to a more detailed study of $\mathcal{M}$.

The techniques for the study of such quotient spaces (which are called "flag manifolds") are well developed.[7,8] We proceed here by use of the theory of loop groups as a guide.[6,7] We show next that $\mathcal{M}$ is a homogeneous Kähler manifold which admits a two-parameter family of homogeneous Kähler metrics.

Since $\mathcal{M}$ is a coset space, DiffS^1 acts transitively by left translations on $\mathcal{M}$. All calculations on $\mathcal{M}$, therefore, can be reduced to algebraic calculations on the tangent space to $\mathcal{M}$ at a fixed point (the origin). The tangent space to $\mathcal{M}$ at the origin can be understood in terms of the Lie algebra, DiffS^1. Since DiffS^1 acts on $\mathcal{M}$, there are vector fields on $\mathcal{M}$ satisfying

$$[L_m,L_n]=(m-n)L_{m+n},\quad m,n\in Z,\qquad \bar{L}_m=L_{-m}. \tag{5}$$

(This is in fact the complexification of the Lie algebra of DiffS^1. Even though DiffS^1 is semi-simple, it has no invariant metric, since the Killing-Cartan metric diverges.) We can identify a tangent vector v to M at the origin as a linear combination of L_m's for $m\neq 0$:

$$v=\sum_{m\neq 0} v_m L_m,\quad \bar{v}_m=v_{-m}. \tag{6}$$

A complex structure at this point is defined by

$$(\tilde{J}v)=-i\sum_{m\neq 0}\text{sgn}(m)v_m L_m. \tag{7}$$

Given this, $\tilde{J}$ is defined at all points of $\mathcal{M}$ by left translation. To show that this complex structure is integrable, we must show that the commutator of two vectors of type (1,0) is also of type (1,0). At the origin, a vector of type (1,0) is a linear combination of L_m's for $m>0$. It is clear that these vectors are closed under commutation. [Since the vector fields L_m are themselves not left invariant, L_m with $m>0$ are not of type (1,0) away from the origin. It is very complicated and unnecessary to find an explicit expression for $\tilde{J}$ away from the origin.]

We have established, then, that $\mathcal{M}$ is a complex manifold. Homogeneous Kähler forms ω on $\mathcal{M}$ are determined by their value at the origin. The condition that ω be closed implies that

$$(d\omega)(L_m,L_n,L_p)=-\omega([L_m,L_n],L_p)+\text{cyclic permutations}=0. \tag{8}$$

It is well known[12] that the most general solution to this algebraic equation is

$$\omega(L_m,L_n)=(am^3+bm)\delta_{m,-n}. \tag{9}$$

For ω to be invertible, either $a=0$, $b\neq 0$, or $a\neq 0$, $-b/a\neq n^2$ with $n\in Z$.

Thus, we have found the most general homogeneous Kähler metric on $\mathcal{M}$. The case $a=0$ has been considered as an inner product on the algebra in the literature,[5,9] but we will see that this is a poor choice in geometry.

We compute now the Riemann tensor of this manifold. Again, all computations can be reduced to algebraic ones at the origin.[7,8] We find

$$R(L_{-m},L_n)L_{-p}=\left\{-\Theta(p-m)(p+m)^2\left[\frac{a(p-m)^3+b(p-m)}{ap^3+bp}\right]+(p+2m)^2\frac{ap^3+bp}{a(p+m)^3+b(p+m)}-2mp\right\}\delta_{m,n}\delta^q_p L_{-q}, \tag{10}$$

where Θ is the Heaviside step function. Of particular interest in Kähler geometry is the Ricci form, $\mathrm{Ric}(L_m,L_n)$, which is given by a trace over p of R. On an infinite-dimensional manifold the trace may not, in general, converge, and the Ricci form may not then exist. We find here that for $a\neq 0$, the Ricci form is well-defined and given by

$$\mathrm{Ric}(L_{-m},L_n)=(-\tfrac{26}{12}m^3+\tfrac{1}{6}m)\delta_{m,n}. \tag{11}$$

This is our first important result. For $a=0$, the Ricci form is divergent and hence ill defined. The Ricci form is independent of a and b. The coefficient of m is easily shown to be $\frac{1}{6}$ for $b=0$, but is in fact independent of a and b.[13] The fact that the sum of the diagonal elements of the curvature tensor converges is the consequence of a remarkable set of cancellations—no regularization is used in the calculation of the Ricci form.

Note particularly the appearance of -26 in the expression for the Ricci form. $\frac{1}{12}m^3\delta_{m,n}$ is a generator of the second cohomology[12] of $\mathcal{M}$. We have found that the Ricci form contains -26 copies of this generator. In finite dimensions, the Ricci form represents the first Chern class of the tangent bundle. With the appropriate generalization to infinite dimensions, we can view the Ricci form as the first Chern class of $\mathcal{M}$. ($\mathrm{Diff}S^1/S^1$ is contractible, so that we would expect its cohomologies to vanish. But we are dealing with the cohomology of forms invariant under $\mathrm{Diff}S^1$, which is nontrivial. We are not aware, however, of a rigorous definition of Chern classes for the type of manifold we are considering.) We note again that the cohomology class of the Ricci form is independent of the choice of metric, as it should be. An intuitive explanation for the negative sign of the curvature is that $\mathcal{M}$ contains $\mathrm{SL}(2,R)/\mathrm{U}(1)$, which is the Bolyai-Lobachevsky space with negative curvature.

The nonzero ghost modes can be identified with vector fields on the manifold $\mathcal{M}$. $\bar{\Gamma}$ then represents the ground state of the ghosts. The Ricci tensor we have computed has a special significance in this context. To be able to specify a reparametrization-invariant ghost vacuum there must exist a covariantly constant tensor density ($\bar{\Gamma}$, or the anticanonical line bundle, which represents the vacuum). This requires the Ricci tensor to vanish. In our case, we see then that no such invariant vacuum state exists. In fact, the Ricci form of a Kähler manifold is the curvature of the covariant exterior derivative on densities, $\alpha\in\bar{\Gamma}$,

$$\nabla^2\alpha=\mathrm{Ric}\wedge\alpha. \tag{12}$$

To resolve the present obstruction we can consider the product of $\bar{\Gamma}$ with some other vector bundle whose curvature is $+26$. Such a vector bundle is provided by open-string theory. To see this, consider the bosonic Fock space $\mathcal{B}$ of open strings in d dimensions as the space of analytic functions on $\mathcal{L}R^{d-1,1}$. The inner product of two functions in loop space is defined as

$$\langle f,g\rangle_0=\int dx_0\prod_{n=1}^{\infty}dx_n\,d\bar{x}_n\bar{f}ge^{-K_0}, \tag{13}$$

where K_0 is the flat Kähler potential defined earlier. It may be verified that this agrees with the usual inner product.[1] (Since $R^{d-1,1}$ has a Lorentzian metric, K is not positive. So the integral has to be defined by an analytic continuation. As a result, the norm on $\mathcal{B}$ is not positive. But this is the usual indefinite norm of covariant quantization.)

Now the definition of $\mathcal{B}$ depends, as usual, on the choice of the complex structure J and the Kähler potential K on $\mathcal{L}R^{d-1,1}$. Let us hold K fixed at K_0 and construct a holomorphic vector bundle[8] Y by varying J all over $\mathcal{M}$. At each point in $\mathcal{M}$ we attach the Fock space constructed from the complex structure defined by that point. $\mathcal{B}$ carries the well-known projective representation of $\mathrm{Diff}S^1$ through the Virasoro operators. There is a natural holomorphic connection on Y that leaves the inner product in $\mathcal{B}$ invariant.[14] We can calculate the curvature (field strength) of this connection by methods similar to those we used previously.[11] We find that the curvature of the covariant derivative in Y is

$$F_0(L_m,L_{-n})=(\tfrac{1}{12}dm^3-\beta m)\delta_{m,n}1. \tag{14}$$

It is to be stressed that the curvature matrix $F_0(L_m,L_{-n})$ is proportional to the identity matrix (this is not true for the tangent bundle of $\mathcal{M}$). We note that this curvature is precisely the contribution to the Virasoro commutation relations from the anomaly. β is determined by the ordering prescription for the Virasoro operators on $\mathcal{B}$.

Now the product bundle Z over $\mathcal{M}$ with fiber $\mathcal{B}\times\bar{\Gamma}$ has a covariant derivative D_0 with curvature

$$\tilde{R}_0(L_m,L_{-n})=D_0^2(L_m,L_{-n})=\left[\frac{d-26}{12}m^3(\tfrac{1}{6}-\beta)m\right]\delta_{m,n}. \tag{15}$$

For $d=26$ and $\beta=\frac{1}{6}$ we see that

$$\tilde{R}_0(L_m,L_{-n})=0. \tag{16}$$

We have an interpretation, then, of the anomaly cancellation of string theory as the vanishing of curvature.[15,16] In the critical dimension we see that one can define a reparametrization-invariant vacuum for the full Hilbert space of bosons and fermions (ghosts). The vacuum is provided by the covariantly constant zero-form, even though such a vacuum cannot be defined for the bosons or ghosts separately.

What we require for a theory of closed strings is an equation of motion for the Kähler potential in loop space. We note that the only place where the background geometry of $\mathcal{L}R^{d-1,1}$ entered was in the definition of the inner product in $\mathcal{B}$. Let us now generalize the Kähler geometry of $\mathcal{L}R^{d-1,1}$ to one with an arbitrary Kähler potential $K(x_0,x_n,\bar{x}_n)$. Actually, we are thinking of $\mathcal{L}R^{d-1,1}$ as a family of complex manifolds, $\mathcal{L}R^{d-1,1}=\Omega R^{d-1,1}\times R^{d-1,1}$. $K(x_0,x_n,\bar{x}_n)$ is a family of Kähler potentials on $\Omega R^{d-1,1}$. The inner product on $\mathcal{B}$ is now defined as

$$\langle f,g\rangle_K=\int dx_0\prod_{n=1}^{\infty}dx_n\,d\bar{x}_n\,\det\omega e^{-K}\bar{f}g, \qquad (17)$$

where $\omega=\partial\bar{\partial}K$ is the Kähler form. This in turn changes the definition of D, the covariant derivative on Z, and its field strength. Let $\bar{R}_K$ be the curvature associated with K. For $K=K_0$, we have already computed this curvature. The natural generalization of the Ricci tensor is the curvature of the vacuum space of the combined Bose-Fermi system $V\times\bar{\Gamma}$, where $V\subset\mathcal{B}$ is the subspace of constant functions on $\Omega R^{d-1,1}$. Consider then the subbundle of Z with fiber $V\times\bar{\Gamma}$. The curvature of this line bundle (a generalized Ricci form) will be donated Ric_K. We postulate that the equation of motion of closed-string theory is that

$$\mathrm{Ric}_K=0. \qquad (18)$$

Our earlier results show that $K=K_0$ (flat space) is a solution of this equation in the critical dimension $d=26$. Interactions arise because the curvature Ric_K depends nonlinearly on K. This is analogous to Einstein's equations. It is now of great interest to find new solutions to these equations.

The next obvious step in the program we have outlined is to find the solution to our equation of motion for small fluctuations about flat space.

The idea of considering the space of all complex structures is reminiscent of twistor theory.[14,17] An approach to string theory also based on complex geometry has been proposed by Friedan and Shenker.[18] It is of interest to know the relation between the "universal Teichmüller space" with which they work and $\mathrm{Diff}S^1/S^1$.

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