Massless Bosonic Free Fields

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The covariant description of massless bosonic free particles in space-time of any dimensionality which carry arbitrary representations of the Lorentz group is presented.

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The covariant description of bosonic free particles carrying arbitrary representations of the Lorentz group is a rather old problem which started with the classical work of Fierz and Pauli[1] in which the representation[2] (2) of a massive particle was studied. Until last year only the descriptions of a few types of representations were known. These types were the representations (n) and $(1,1,\ldots,1)$ for both massless[3,4] and massive[5] particles, and the mixed-symmetry ones, (2,1), (2,1,1), (2,2), and $(n,1)$, also for both massless and massive cases.[6,7] A good review of the progress made up to 1980 can be found in the work of Curtright.[8] More recent papers[9] contain some related work on the subject. The situation changed drastically since last year. With the advent of string field theory[10] the covariant description of any representation for the massive case was obtained. From the well-known fact that covariant massless and massive descriptions are related by compactification of one of the dimensions one could in principle obtain also the complete description for the massless case. However, the kind of *anticompactification* that should be involved in such a process is a tedious and inelegant procedure which does not teach us anything about the rich physics contained in the description of massless particles. One would certainly prefer a description based on the principle of gauge invariance. The importance of gauge invariance in dealing with the covariant formulation of particles carrying arbitrary representations of the Lorentz group was first pointed out by Curtright in Ref. 3.

Recent work[11,12] has shown an increasing interest in the covariant description of free particles carrying arbitrary representations by use of the principle of gauge invariance. In Ref. 11 the complete covariant description for representations of the type $(2,1,1,\ldots,1)$ was presented. In Ref. 12 the principle of gauge invariance was stated for any representation, the ghost content was identified with complete generality except for trace conditions, and the complete description was presented for representations of the type $(2,2,\ldots,2,1,1,\ldots,1)$. The methods presented in Ref. 12 cannot be extended easily to the general case, because for representations corresponding to Young tableau (YT) with more than two columns gauge invariance implies certain trace conditions on the gauge parameters. In addition, for representations corresponding to YT with more than three columns the field itself must obey certain double-trace conditions in order to have the right number of physical degrees of freedom. These facts are known from the description of the completely symmetric cases.[3,4] Their necessity when dealing with mixed symmetries follows from simple counting of the ghost content present in Ref. 12.

In this note, I present a solution to this long-standing problem which is inspired by methods utilized in string field theories. I suspect that this formulation may be a manifestation of the existence of a new class of theories which involve interactions among fields of any representation. The simplicity and uniqueness of the formulation is very appealing. In the past years we have learned that it seems that in order to formulate consistent interacting theories involving higher-spin particles one needs to introduce all the representations. The fact that there exists a compact form to deal with all the bosonic representations may be an indication that what is presented in this note is an operator representation of the free part of a new class of theories which involve massless particles of any spin.

Before presenting my analysis I will review the principle of gauge invariance as stated in Ref. 12. Given an arbitrary representation of the little group of a massless particle in d dimensions, $(a_1,a_2,\ldots,a_M)$, I will postulate that the covariant description can be formulated with a field A whose space-time indices have the structure of the YT corresponding to the irreducible representation of $GL(d)$ $[a_1,a_2,\ldots,a_M]$. Additionally, I postulate that the action of this field is invariant under gauge transformations whose gauge parameters have an index structure corresponding to all the YT that one can make by removing one box from the YT of the original field. In this formalism all the representations of the little group which are described are irreducible except the self-associated ones, which split into irreducible self-dual and anti-self-dual parts.

The gauge transformations postulated above can be expressed very simply by use of the context of bosonic string fields. Consider a vector field $|\mathcal{A}(x)\rangle$ in the Fock space spanned by a set of N covariant oscillators a_n^ν which satisfy

$$[a_m^{\dagger\mu},a_n^\nu]=\delta_{mn}\eta^{\mu\nu}, \tag{1}$$

where $\eta^{\mu\nu}=(1,-1,-1,\ldots,-1)$ with $d-1$ entries -1. The vector field $|\mathcal{A}(x)\rangle$ represents a collection of local fields which are just the coefficients of its expansion:

$$|\mathcal{A}(x)\rangle=\phi(x)|\,\rangle+A_\mu^m(x)a_m^{\dagger\mu}|\,\rangle+\tfrac{1}{2}G_{\mu\nu}^{mn}(x)a_m^{\dagger(\mu}a_n^{\dagger\nu)}|\,\rangle+\tfrac{1}{2}A_{\mu\nu}^{mn}(x)a_m^{\dagger[\mu}a_n^{\dagger\nu]}|\,\rangle+\ldots, \tag{2}$$

where $|\,\rangle$ represents the vacuum. For $N\geq d$ all the representations appear in the expansion (2). In this framework gauge transformations can be formulated in the following way. Consider N collections of gauge parameters represented by the vectors $|\mathcal{E}_n(x)\rangle$, $1\leq n\leq N$. The gauge parameters are the coefficients of its corresponding expansions:

$$|\mathcal{E}_n(x)\rangle=\varepsilon^n(x)|\,\rangle+\varepsilon_\mu^{nm}(x)a_m^{\dagger\mu}|\,\rangle+\ldots. \tag{3}$$

Gauge transformations, as postulated above, are generated by our defining the transformation of the vector field $|\mathcal{A}(x)\rangle$ by

$$\delta|\mathcal{A}(x)\rangle=a_m^{\dagger\mu}\partial_\mu|\mathcal{E}_m(x)\rangle. \tag{4}$$

Identifying the coefficients in the expansions of each side of Eq. (4) one can observe that in fact the local fields of Eq. (2) transform properly under gauge transformations.

Our next task is to construct the vector-field equation which $|\mathcal{A}(x)\rangle$ must satisfy. A discussion based on the field equations appears to be much simpler. To find the vector-field equation involves the construction of an operator $\mathcal{O}$ such that

$$\mathcal{O}|\mathcal{A}(x)>=0, \tag{5}$$

with the following properties: (a) $\mathcal{O}$ must be a second-order differential operator with respect to the space-time coordinates x^μ; (b) if we normally order the terms entering into $\mathcal{O}$, each term must have the same number of creation and annihilation operators; (c) the terms entering into $\mathcal{O}$ cannot be operators that involve the Laplacian and one or more pairs of creation and annihilation operators; (d) the vector-field equation (5) must be invariant under the gauge transformation (4). Property (b) ensures that there is no mixing between different representations. Property (c) avoids terms which are superfluous. Suppose that we go along with those terms and we work out the field equations of the local fields. One would have at least two types of terms originating from those unwanted operators: terms with the Laplacian acting on traces of the local fields and terms with the Laplacian acting on the local field itself. The terms of the second type could be combined with the terms resulting from the operator with no creation and annihilation operators, the simple Laplacian (that necessarily has to be there if one wants to describe scalar fields properly). The terms of the first type could be expressed in terms of operators which do not involve the Laplacian by use of the field equations: Take a trace of the field equation, solve for the Laplacian acting on that trace, and substitute it back in the full field equation. As I will show below this condition makes the formulation extremely simple. This condition is responsible for the nonhermiticity of $\mathcal{O}$. The four properties above determine $\mathcal{O}$ *uniquely* up to overall normalization. I show now how this construction is made.

Let us classify the possible normal-ordered terms entering into $\mathcal{O}$ by the number of creation operators (that I will call degree) that it contains and then by the inequivalent ways that the contraction of their indices can be arranged. In this classification, there are one operator of degree zero, the Laplacian, $\Box$; one of degree one, $a_n^{\dagger\alpha}a_n^\beta\partial_\alpha\partial_\beta$; seven of degree two,

$$\begin{gathered}a_m^{\dagger\alpha}a_n^{\dagger\beta}a_m^\gamma a_{n,\gamma}\partial_\alpha\partial_\beta,\quad a_m^{\dagger\alpha}a_n^{\dagger\gamma}a_m^\beta a_{n,\gamma}\partial_\alpha\partial_\beta,\\ a_m^{\dagger\alpha}a_m^{\dagger\beta}a_n^\gamma a_{n,\gamma}\partial_\alpha\partial_\beta,\quad a_m^{\dagger\alpha}a_n^{\dagger\gamma}a_{m,\gamma}a_n^\beta\partial_\alpha\partial_\beta,\\ a_m^{\dagger\gamma}a_{n,\gamma}^\dagger a_m^\alpha a_n^\beta\partial_\alpha\partial_\beta,\quad a_m^{\dagger\alpha}a_m^{\dagger\gamma}a_n^\beta a_{n,\gamma}\partial_\alpha\partial_\beta,\\ a_m^{\dagger\gamma}a_{m,\gamma}^\dagger a_n^\alpha a_n^\beta\partial_\alpha\partial_\beta;\end{gathered} \tag{6}$$

29 of degree three, which we do not list; etc.

The first important observation to be made is the fact that after a gauge transformation in (5) each term of $\mathcal{O}$ gets one extra creation operator. When this operator is commuted to the left to obtain normal-ordered terms two types of operators are generated. Suppose that we started with an operator in $\mathcal{O}$ of degree n. This operator generates normal-ordered operators with n creation operators and $n-1$ annihilation operators and a remainder which consists of $n+1$ creation operators and n annihilation operators. Clearly, for the case of degree zero (the Laplacian) there is a remainder only. If we perform a gauge transformation in (5), cancellations can occur if the coefficients of the terms entering $\mathcal{O}$ are arranged in such a way that the remainders of the operators of degree n are canceled by the nonremainders originating from the operators of degree $n+1$. We will analyze how the coefficients can be arranged starting with the operators of lowest degree, i.e., the Laplacian $\Box$. This fixes the overall normalization. To cancel the remainder generated from this operator one needs to introduce the operator of degree one, $a_n^{\dagger\alpha}a_n^\beta\partial_\alpha\partial_\beta$, with coefficient 1. To cancel the remainder generated from this operator one needs to introduce the seven operators of degree two listed in (6) with arbitrary coefficients. The nonremainder terms generated from these operators after the transformation (4) are the seven inequivalent ones (which are in fact the complete set which could appear)

$$a_m^{\dagger\alpha}a_n^{\dagger\beta}a_m^{\gamma}\partial_\alpha\partial_\beta\partial_\gamma,\quad a_n^{\dagger\alpha}a_m^{\dagger\gamma}a_{m,\gamma}\Box\partial_\alpha,$$
$$a_m^{\dagger\alpha}a_m^{\dagger\beta}a_n^{\gamma}\partial_\alpha\partial_\beta\partial_\gamma,\quad a_m^{\dagger\gamma}a_{m,\gamma}^{\dagger}a_n^{\alpha}\Box\partial_\alpha,$$
$$a_m^{\dagger\alpha}a_m^{\dagger\gamma}a_{n,\gamma}\Box\partial_\alpha,\quad a_m^{\dagger\gamma}a_{n,\gamma}^{\dagger}a_m^{\alpha}\Box\partial_\alpha, \tag{7}$$
$$a_m^{\dagger\alpha}a_n^{\dagger\gamma}a_{m,\gamma}\Box\partial_\alpha.$$

To cancel the remainder from the operator of degree one, it is necessary to arrange coefficients in such a way that only the first operator in (7) survives with coefficient -1. This generates an inhomogeneous linear system of equations for the coefficients of the operators listed in (6). One can verify that the system has a unique solution. The solution is such that only the first operator in (6) enters into $\mathcal{O}$, with coefficient $\frac{1}{2}$. The next step is to study the possible cancellation of the remainder from the only operator of degree two that enters $\mathcal{O}$. This remainder is

$$\tfrac{1}{2}\,a_m^{\dagger\alpha}a_n^{\dagger\beta}a_p^{\dagger\gamma}a_m^{\delta}a_{n,\delta}\partial_\alpha\partial_\beta\partial_\gamma. \tag{8}$$

To cancel (8) one needs to proceed with the same analysis applied to the operators of degree three. As was mentioned earlier, there are 29 operators of this degree. The variation of (5) under the gauge transformation (4) produces 40 inequivalent operators from the ones of degree three and it turns out that there is no way to cancel the operators of degree two. Gauge invariance forces us to stop here and to demand the following constraint for the gauge parameters:

$$\eta_{\alpha\beta}a^{\alpha}_{(m}a^{\beta}_{n}\,|\,\mathscr{E}_{p)}(x)\rangle=0, \tag{9}$$

where the symmetrization comprises the three indices m, n, and p. Condition (9) involves traces of the local fields which is what one expects to find. In summary, gauge invariance forces the gauge parameters to be constrained according to (9) and singles out the operator $\mathcal{O}$ to be

$$\mathcal{O}=\Box+a_m^{\dagger\alpha}a_m^{\beta}\partial_\alpha\partial_\beta+\tfrac{1}{2}\,a_m^{\dagger\alpha}a_n^{\dagger\beta}a_m^{\gamma}a_{n,\gamma}\partial_\alpha\partial_\beta. \tag{10}$$

In describing massless particles one has to take into account the different generations of gauge invariance that the theory possesses. In Becchi-Rouet-Stora-Tyutin language this corresponds to the different generations of ghosts. More than one generation is necessary when one treats representations which correspond to YT with more than two rows. In the formulation introduced in this note there is a very simple way to deal with higher-generation gauge parameters. The gauge transformation of the first-generation gauge parameters is given by

$$\delta\,|\,\mathscr{E}_n(x)\rangle=a_m^{\dagger\mu}\partial_\mu\,|\,\mathscr{E}_{nm}(x)\rangle, \tag{11}$$

where $|\,\mathscr{E}_{nm}(x)\rangle$ are a set of vectors which constitutes a collection of second-generation gauge parameters. The vectors $|\,\mathscr{E}_{nm}(x)\rangle$ are such that $|\,\mathscr{A}(x)\rangle$ remains invariant under a double gauge transformation. From (4) and (11) this implies that the vectors $|\,\mathscr{E}_{nm}(x)\rangle$ must satisfy the constraint

$$a_m^{\dagger(\mu}a_n^{\dagger\nu)}\,|\,\mathscr{E}_{nm}(x)\rangle=0, \tag{12}$$

which tells us that not all the components in $|\,\mathscr{E}_{nm}(x)\rangle$ are independent. This fact is already known from the analysis performed in Ref. 12 where the classification of which representations appear in each generation was carried out. One can verify that, in fact, both approaches give the same answer. However, the formulation presented in this note gives us more information; one can also obtain the trace conditions satisfied by the representations of each generation. These trace conditions appear in this context as consistency conditions from the fact that $|\,\mathscr{E}_n(x)\rangle$ is constrained according to (9). From (9) and (11) one finds that $|\,\mathscr{E}_{nm}(x)\rangle$ must also satisfy

$$\eta_{\alpha\beta}a^{\alpha}_{(m}a^{\beta}_{n}a^{\dagger\mu}_{|q|}\,|\,\mathscr{E}_{p)q}(x)\rangle=0, \tag{13}$$

where $|q|$ means that the index q must be excluded from the symmetrization. Similarly one can analyze the conditions satisfied by the next generations of gauge parameters. For example, for the third one

$$\delta\,|\,\mathscr{E}_{nm}(x)\rangle=a_p^{\dagger\mu}\partial_\mu\,|\,\mathscr{E}_{nmp}(x)\rangle,$$
$$a_m^{\dagger(\mu}a_p^{\dagger\nu)}\,|\,\mathscr{E}_{nmp}(x)\rangle=0,$$
$$a_m^{\dagger(\mu}a_n^{\dagger\nu)}a_p^{\dagger\sigma}\,|\,\mathscr{E}_{nmp}(x)\rangle=0, \tag{14}$$
$$\eta_{\alpha\beta}a^{\alpha}_{(m}a^{\beta}_{n}a^{\dagger\mu}_{|q}a^{\dagger\nu}_{r|}\,|\,\mathscr{E}_{p)qr}(x)\rangle=0.$$

So far we have found a classical theory involving bosonic fields transforming according to all the representations of GL(d) with a series of generations of gauge invariance. This theory may or may not have anything to do with massless particles. Its uniqueness makes it very appealing to think that it corresponds to the description of massless particles. However, one needs to prove that the physical degrees of freedom of the theory do indeed transform as representations of the little group SO($d-2$). One way to check this is to analyze the field equation and gauge invariance associated to each of the local fields. We have verified that the counting of degrees of freedom is correct for representations whose YT have at most three columns. However, for representations with four or more columns one has too many physical degrees of freedom. This is not surprising since we know that certain double-trace conditions must be imposed on the local fields for these cases. In this context, the generalization of that condition that seems to work is

$$L_{mnpq}\,|\,\mathscr{A}(x)\rangle=0, \tag{15}$$

where

$$L_{mnpq} \equiv \eta_{\alpha\beta}\eta_{\gamma\delta} a^{\alpha}_{(m} a^{\beta}_{n} a^{\gamma}_{p} a^{\delta}_{q)}. \qquad (16)$$

As in previous analyses[3,4,7,12] this constraint is not a consequence of gauge invariance; it is an additional constraint that must be imposed on the vector field $|\mathcal{A}(x)\rangle$ itself. Constraint (15) is consistent with the vector-field equation (5) since, after some algebra, one finds

$$[L_{mnpq},\mathcal{O}] = 2L_{mnpq}\Box - \tfrac{1}{6} a_r^{\dagger\mu} a^{\nu}_{(m} L_{npq)r} \partial_\mu \partial_\nu. \qquad (17)$$

This equation shows that the double traces removed from $|\mathcal{A}(x)\rangle$ when constraint (15) is imposed are effectively decoupled in the wave equation (5). It is worth remarking that the symmetrization in the roman indices in (16) is essential in order to have a commutator which is proportional to the L_{mnpq} operators. Furthermore, explicit analysis[13] of the counting in the description of particular representations shows that such a symmetrization is responsible for obtaining the right number of physical degrees of freedom. It can also be proved that constraint (15) does not imply additional conditions for the gauge parameters. I have analyzed many of the representations and the conjecture that (5), (9), (10), (15), and (16) constitute the full description seems to be correct (details will be shown elsewhere[13]).

This work opens a variety of investigations. First of all, one would like to find a Lagrangean formulation and to *solve* constraint (15). Certainly, the extension of this work to include fermions (with and without supersymmetry) has to be carried out. Gauge-fixing procedures, the Becchi-Rouet-Stora-Tyutin formulation, and quantization have also to be analyzed. Finally, one should try to formulate interacting theories in this context. One way to proceed in this direction could involve the search for a non-Abelian form of the gauge transformation (4).

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[1]M. Fierz and W. Pauli, Proc. Roy. Soc. London, Ser. A **173**, 211 (1939).

[2]Representations of the little group of a massless (massive) particle in d dimensions, SO$(d-2)$ (SO$(d-1)$), are denoted by the usual Young tableau [see for example: M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading, MA, 1962)] notation $(a_1, a_2, \dots, a_M)$, where a_i indicates the number of boxes in the ith row. Similarly, the irreducible representations of GL(d) are labeled by the standard notation $[a_1, a_2, \dots, a_M]$.

[3]T. Curtright, Phys. Lett. **85B**, 219 (1979).

[4]C. Frondsal, Phys. Rev. **161**, 3624 (1978); B. de Wit and D. Z. Freedman, Phys. Rev. D **21**, 358 (1980).

[5]S. J. Chang, Phys. Rev. **161**, 1308 (1967); L. P. S. Singh and C. R. Hagen, Phys. Rev. D **9**, 898 (1974).

[6]T. Curtright and P. G. O. Freund, Nucl. Phys. **B172**, 413 (1980).

[7]T. Curtright, Phys. Lett. **165B**, 304 (1985).

[8]T. Curtright, in *High Energy Physics—1980,* edited by L. Durand and L. G. Pondrom, AIP Conference Proceedings No. 68 (American Institute of Physics, New York, 1981), pp. 985–988.

[9]T. Curtright, University of Florida Report No. UFTP-82-22 (unpublished); G. R. E. Black and G. B. Wybourne, J. Phys. A **16**, 2405 (1983); R. Delbourgo and P. Jarvis, J. Phys. A **16**, L275 (1983).

[10]W. Siegel, Phys. Lett. **149B**, 157, 162 (1984), and **151B**, 391, 396 (1985); W. Siegel and B. Zweibach, Nucl. Phys. **B263**, 105 (1986).

[11]C. S. Aulakh, I. G. Koh, and S. Ouvry, Phys. Lett. **173B**, 284 (1986).

[12]J. M. F. Labastida and T. R. Morris, Phys. Lett. **180B**, 101 (1986).

[13]J. M. F. Labastida, to be published.