Anomalous Transport and the Coupling of Plasma Diffusion and Heat Flow

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Anomalous transport has been observed in heat-pulse tokamak experiments in the form of faster thermal diffusion than expected from thermal conductivity alone. The anomaly is resolved by correct treatment of time-dependent transport, coupling heat flow and plasma diffusion. The diffusion rates are exhibited as invariant eigenvalues appearing in the transport model. We show how these eigenvalues couple the basic transport processes. The thermal diffusion rate is not determined by thermal conductivity, since it is not an eigenvalue with temperature as eigenmode.

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Temperature perturbation experiments in tokamak plasmas exhibit anomalous heat transport in the form of faster thermal diffusion than predicted by electron thermal conductivity. Fredrickson *et al.*[1] have estimated the "effective" thermal conductivity by analyzing experimental and numerical data. Their analysis finds effective conductivity for a heat pulse to be about 1.5 times larger than the heat conductivity put into numerical codes. Goedheer[2] tried to resolve this anomaly by choosing different ways of estimating effective thermal conductivity.

Grad[3] has attributed the apparent anomaly to an incomplete understanding of the coupling of plasma diffusion and heat flow. Grad[4] extracted eigenvalues from the transport equations and found that the transport is governed by eigenvalues which are functions of both the resistivity η, and the thermal conductivity χ. Since thermal conductivity is not an eigenvalue there is automatic coupling of temperature to mass diffusion. Here, we find a new set of variables, viz., approximate eigenmodes, which have those eigenvalues as their natural time constants. We can numerically excite each eigenmode separately and verify that all relevant profiles follow the decay rate predicted by the eigenvalues; these rates can differ substantially from those predicted by simple conductivity or resistivity considerations.

We consider the Grad-Hogan[5] model of slow evolution obtained from the dissipative magnetohydrodynamic (MHD) equations by dropping of the inertia term. In this scaling of the MHD equations, the transport evolves through successive equilibria established by the pressure balance $\nabla p = \mathbf{J} \times \mathbf{B}$.

The transport problem has been cast into an invariant (also known as alternating dimension) formulation by Grad.[4] The key step is to eliminate the velocity by taking microcanonical volume averages on each flux surface and introducing suitable adiabatic dependent and independent variables. The prototype system takes the following form:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(A_{ij}\frac{\partial u_j}{\partial x}\right) + \tilde{R}_i.$$

Here we choose $u_1 = \nu$, $u_2 = \xi$, $u_3 = \sigma$, and $x = \chi$. Thus we obtain a set of one-dimensional evolution equations for rotation number ν and mass density ξ with respect to toroidal flux χ (also denoted by ψ_1) and entropy density σ, where for pressure p, mass density ρ, and ratio of specific heats γ, $p = \exp(\sigma)\rho^\gamma$. The evolution equations are

$$\frac{D\nu}{Dt} = \frac{\partial}{\partial \chi}\left(\lambda_0 \frac{\partial \nu}{\partial x}\right) + \frac{\partial}{\partial \chi}(\eta_\parallel A), \quad \frac{D\xi}{Dt} = \frac{\partial}{\partial \chi}\left[\eta_\parallel \xi\left(\varepsilon_\eta R_1 \frac{\partial p}{\partial \chi} + \phi_2 \frac{\partial \phi_2}{\partial \chi}\right)\right],$$

$$\frac{D\sigma}{Dt} = \frac{\gamma - 1}{p}\left[a\frac{\partial}{\partial \chi}\left(a\xi^2 \frac{\kappa_1 \varepsilon_\kappa S_1}{T^{1/2}} \frac{\partial T}{\partial \chi}\right) + \eta J^2 + \eta_\parallel\left(\varepsilon_\eta R_1 \frac{\partial p}{\partial \chi} + \phi_2 \frac{\partial \phi_2}{\partial \chi}\right)\frac{\partial \sigma}{\partial \chi}\right],$$

supplemented by the ordinary differential equation for average pressure balance

$$[(\gamma p + B^2)/a]\partial a/\partial \chi + a\phi_2 \partial \nu/\partial \chi + (\gamma p/\xi)\partial \xi/\partial \chi + p\,\partial \sigma/\partial \chi + a^2 C = 0,$$

with magnetic field B, current density J, and temperature T, where $Du_i/Dt = \partial u_i(\chi,t)/\partial t$. Resistivity parallel to B is $\eta_\parallel$, $a = d\chi/dV$, V is volume inside a flux surface, Λ_{ij} is the element of the inductance matrix relating the toroidal and poloidal magnetomotive forces ϕ_j ($j=1,2$) to the differential fluxes ψ_i' (prime denotes differentiation with respect to V), $\lambda_0=\eta_\parallel a^2\det(\Lambda)$, $A=a^2(l_{11}+vl_{12}+v^2l_{22})$ with

$$l_{11}=\Lambda_{11}\,\partial\Lambda_{12}/\partial\chi-\Lambda_{12}\,\partial\Lambda_{11}/\partial\chi,\quad l_{12}=\Lambda_{11}\,\partial\Lambda_{22}/\partial\chi-\Lambda_{22}\,\partial\Lambda_{11}/\partial\chi,\quad l_{22}=\Lambda_{12}\,\partial\Lambda_{22}/\partial\chi-\Lambda_{22}\,\partial\Lambda_{12}/\partial\chi,$$

$$C=\partial\Lambda_{11}/\partial\chi+2v\,\partial\Lambda_{12}/\partial\chi+v^2\,\partial\Lambda_{22}/\partial\chi,\quad R_1=(\psi_1'^2/p'^2)(\langle J^2\rangle-\langle\mathbf{J}\cdot\mathbf{B}\rangle^2/\langle B^2\rangle),\quad S_1=\langle|\nabla\chi|^2/B^2\rangle.$$

The angular bracket denotes flux averages. Anisotropic Braginskii transport yields the resistivity and thermal conductivity corrections

$$\varepsilon_\eta=1+[(\eta_\perp-\eta_\parallel)/\eta_\parallel]S_1/R,$$

$$\varepsilon_\kappa=0.2+(0.8/S_1)(R_1-\phi_2^2/\langle B^2\rangle);$$

$\eta_\parallel=\eta_1/T^{3/2}$, $\eta_\perp=2\eta_\parallel$; η_1 is a numerical factor.

The above system is truly diffusive: It gives the time derivatives of the dependent variables in terms of second-order spatial derivatives, and thus represents actual transport phenomena. The original system has convective terms besides diffusive terms; consequently, transport phenomena can be obscured by convective effects.

We solve the transport problem by the standard alternating-dimension method. The 3D solution provides the geometric quantities required for the 1D system to progress. We have used several 3D equilibrium codes, in this case the code of Bauer, Betancourt, and Garabedian.[6]

To investigate diffusion we examine the matrix A_{ij} of coefficients of the second derivatives in the 1D evolution equations. The three eigenvalues of A_{ij} are the transport coefficients for the plasma.[4] The eigenvalues are invariant, independent of the choice of dependent or independent variables. Thus the eigenvalues represent true transport coefficients of diffusion of one adiabatic variable with respect to another.[4] In this linearization, the largest eigenvalue λ_0, representing the skin effect, is decoupled from the heat flow and appears explicitly in the v evolution equation. The other eigenvalues $\lambda_\pm$ represent the coupled effects of plasma diffusion and heat flow. From Grad,[4] we have

$$\lambda_0=\eta_\parallel a^2\det(\Lambda),$$

$$\lambda_\pm=\tfrac{1}{2}(\lambda_1+\lambda_2)\pm\tfrac{1}{2}(\lambda_1^2+\lambda_2^2-2\delta\lambda_1\lambda_2)^{1/2},$$

where

$$\delta=[p+(2\gamma^{-1}-1)\langle B^2\rangle]/(p+\langle B^2\rangle).$$

The constants λ_1 and λ_2 separate the explicit contributions from η and κ, the resistivity and thermal conductivity (there is also lower-order nonlinear coupling of everything in the full numerical solution):

$$\lambda_1=[\gamma p\langle B^2\rangle/(\gamma p+\langle B^2\rangle)]\eta_\parallel(\varepsilon_\eta R_1-\phi_2^2/\langle B^2\rangle),$$

$$\lambda_2=(\gamma-1)(p+\langle B^2\rangle)p\kappa_1\varepsilon_\kappa/T^{1/2}(\gamma p+\langle B^2\rangle),$$

where κ_1 is another numerical factor. Only the eigenvalues $\lambda_\pm$ are diffusion rates; λ_1 and λ_2 are parameters. But λ_2 has taken an important role in this Letter because it is the coefficient of the second derivative of T in the 1D T-evolution equation. It compares with χ_e for the cylindrical coordinates of Refs. 1 and 2, where heat transport is called anomalous because χ_e is expected to determine the thermal diffusion rate, although we now know that it cannot, because is it neither an eigenvalue nor is temperature an eigenmode. The form of $\lambda_\pm$ shows that the heat flow and the resistivity are intrinsically coupled. To establish the role of these eigenvalues in transport, we introduce approximate eigenmodes (v,v_+,v_-) transformed from (v,ξ,σ). We introduce a delta-function source (in time and toroidal flux) such that only one of the three pure modes is excited. We solve the 3D transport problem, starting from the background initial state $v=1-0.5y$, $\xi=1-y$, and $T=1-0.5y$, where $y=\chi/\chi_*$, the toroidal flux normalized by its value at the plasma edge. We run until the profiles are approximately equilibrated ($t\cong0.1$ in skin time). Then we excite a particular mode and observe the decay for time scales ranging from $t=0.0001$ to 0.01 in various runs. The duration of the excitation is typically $\frac{1}{50}$ of the total time of observation; the width is $0.02\chi_*$. For each profile we then fit a Gaussian function to the evolving perturbation in order to estimate an approximate diffusion rate λ_u numerically:

$$u=\frac{a_u}{[4\pi\lambda_u(t-t_0)]^{1/2}}\exp\left[-\frac{(y-y_0)^2}{4\lambda_u(t-t_0)}\right].$$

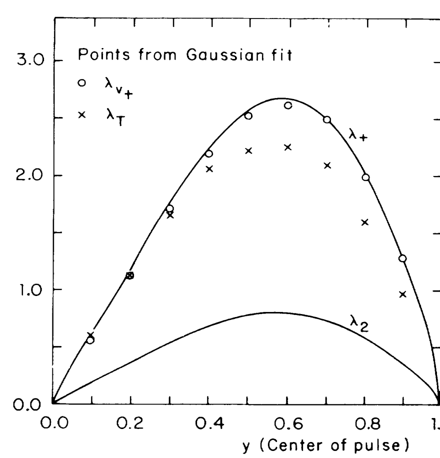

FIG. 1. Comparison of λ_T with the excited pure eigenmode for λ_+ and with λ_2.

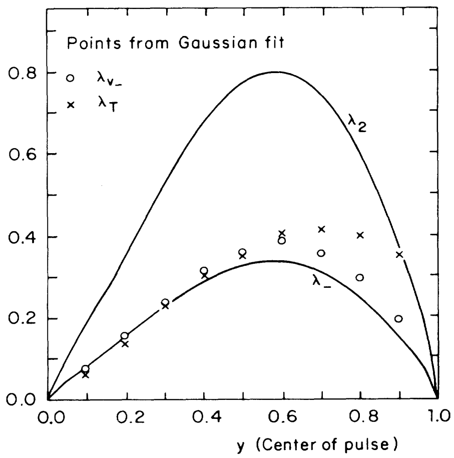

FIG. 2. Comparison of λ_T with the excited pure eigenmode for λ_- and with λ_2.

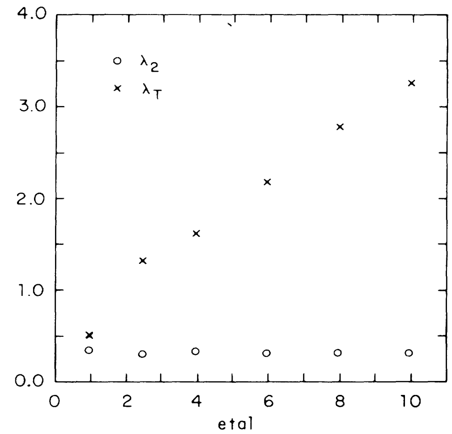

FIG. 3. Comparison of λ_T with λ_2, varying η when only temperature is perturbed.

For each profile, we thus describe the perturbation approximately as the solution of a diffusion equation of the form $\partial u/\partial t = \lambda_u \partial^2 u/\partial y^2$.

The appropriate sources in the various evolution equations are chosen to make the Gaussian fit valid: The pulse must be local and, for a given width, the amplitude must be moderately large to make the second derivatives large. Too large a pulse makes the numerics less accurate and the eigenvalues themselves change by large amounts so that comparison becomes difficult.

We found that the temperature, pressure, and density have almost the same decay rate as λ_+ or λ_- corresponding to the mode excited. In this Letter we concentrate on temperature only. When λ_0 is excited, the v profile decays with time constant λ_0. In that case the other profiles do not have any Gaussian-type perturbations.

Our conclusions are summarized in Figs. 1–3. Figure 1 reports the results of nine runs when the λ_+ mode is excited. The temperature decay rate is near the time scale predicted by the eigenvalue λ_+, which is substantially higher than the value λ_2 predicted by thermal conductivity alone.

Figure 2 reports the results of runs when λ_- is excited. Here we see that the temperature diffusion rate is near λ_-, the eigenvalue of the mode excited, and much lower than λ_2.

The calculated λ's in Figs. 1 and 2 are time independent and unaffected by the geometry.

The most significant result is that the temperature diffusion rate follows λ_+ or λ_- depending upon the mode excited; it does not follow the intuitive rate λ_2. However, if a perturbation in temperature alone is introduced, both eigenmodes are excited. The temperature decay rate cannot be followed by fitting a Gaussian parametrized by a time-independent λ. This is because formally no heat equation can be written for T without ignoring the coupling of the plasma diffusion and heat flow. But numerically one can make such a fit at each instant and estimate an approximate rate λ_T which will now depend on time. In order to make quantitative comparisons with experiments, we require time-dependent data, not available to us as yet.

For runs having temperature perturbations only, the numerically fitted λ_T decreases in time. In Fig. 3 we look at λ_T's from runs at comparable (resistive) times at a fixed location with perturbations that have not yet spread too much and before nonlocal effects become important. The plot shows λ_T for runs with constant κ_1 but varying η_1. The plot also shows the corresponding values of λ_2. For each run, the pulse is centered at $y=0.25$. As known from its algebraic expression, λ_2 is almost independent of η. The deviation of λ_T from λ_2 due to the coupling of κ with η is in the direction reported in Refs. 1 and 2 but, as we have already remarked, this is not an anomaly since λ_2 is not an eigenvalue and temperature is not an eigenmode. Large η_1 for constant κ_1 is equivalent to $\kappa \rightarrow 0$ keeping η_1 constant. Therefore the plot clearly shows that when κ approaches zero we still have thermal diffusion, a result not found in other models.

Note that for high-β plasmas the apparent anomaly will be slight; the size of the apparent anomaly is highest for low-β plasmas.

Enhanced transport due to magnetic and electrostatic fluctuations is also proposed as a possible explanation for the heat-pulse propagation anomaly.[1] Our findings do not eliminate this possibility. We do find that coupling is inherent in transport, an ever present source of apparent anomaly, whatever the other factors that may be at work.

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