Proton Recoil and Radiative Level Shifts

Ganesh Bhatt and Howard Grotch
Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802
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Proton mass corrections to the Lamb shift of hydrogen of order $\alpha(Z\alpha)^5m^2/M$ are evaluated. These contributions are expected new terms beyond those given earlier for recoil effects to the self-energy of a bound electron in the external-field approximation. The new contribution of this order is -0.53 kHz, thus making the complete radiative-recoil correction of this order -2.53 kHz.

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Various aspects of the $2S_{1/2}$-$2P_{1/2}$ Lamb shift in hydrogen have been reviewed in recent years by Lepage and Yennie,[1] Kinoshita and Sapirstein,[2] and Bhatt and Grotch.[3] The comparison of theory and experiment has historically provided one of the most significant low-energy tests of quantum electrodynamics. A numerical calculation by Sapirstein,[4] done for the ground state of hydrogen, provided confirmation of the corresponding work of Mohr, and thus supported Mohr's[5] theoretical Lamb shift, which disagreed with the analytical value given earlier by Erickson.[6] It is now believed that Mohr's value is correct and that the error estimate of Erickson was too optimistic. The most recent experimental values of the Lamb shift are those of Lundeen and Pipkin[7] and Pal'chekov, Sokolov, and Yakovlev.[8]

At the level of present experimental accuracy, the theoretical Lamb shift is very sensitive to the electromagnetic structure and the finite mass of the proton. A relatively new result[9] on the electromagnetic radius of the proton, $\langle r_p^2\rangle^{1/2}=0.862(12)$ fm, is significantly different from the older value[10] [0.805(11) fm]. These values would imply contributions to the theoretical Lamb shift which differ by about 20 kHz. On the other hand, the radiative-recoil contributions of order $\alpha(Z\alpha)^5m^2/M$, the pure-recoil corrections of order $(Z\alpha)^6m^2/M$, and the two-loop nonrecoil binding of order $\alpha^2(Z\alpha)^5m$ could also be as large as 10 kHz. Hence it is necessary to compute these unknown contributions in order to predict the theoretical value with better precision.

Recently, we have considered the modification of the general radiative Lamb-shift calculation technique of Erickson and Yennie[11] to incorporate proton recoil corrections. Results of an analytical evaluation of proton mass corrections to the Lamb shift, by considering the self-energy of an electron in the so-called external-field approximation, have been reported.[3] The leading terms[11] of the order

$$(m^2/M)\alpha(Z\alpha)^4[C_0+C_1\ln(Z\alpha)+C_2\ln(m/\Delta E_n)]$$

were systematically derived and parts of the new radiative-recoil contribution of order $\alpha(Z\alpha)^5m^2/M$ contained in this self-energy were identified. In that paper we indicated that additional contributions of the same order will also arise from corrections to the external-field approximation. In the present paper, we report the result of the calculation of these contributions, which arise by considering the two-photon exchange kernels not included in the previous approximation.

We outline here the important steps of this calculation and postpone the detailed description for a more complete publication. The interaction kernels involving the two-photon exchange are shown in Fig. 1. The radiative photon is on the electron line. The correction on the external line is necessary, as usual, for the consistent cancellation of ultraviolet divergent terms. It is important to note that from the ladder exchange diagrams, the contributions corresponding to the on-shell positive-energy proton are subtracted. These constitute all terms which are already included in Ref. 3.

The calculation follows the standard methods[12] for finding the amplitude for the perturbation in question. In terms of the amplitude, ΔK, the energy-level shift is given by

$$\Delta E_n=\int d^3r\,d^3r'\,\bar{\psi}_n(\mathbf{r}')i\Delta K(\mathbf{r}',\mathbf{r})\psi_n(\mathbf{r}), \tag{1}$$

where $\psi_n(\mathbf{r})$ is the momentum-space solution for the atomic n state of the external-field equation [Eq. (2.9) of Ref. 3].

The construction of ΔK is quite involved. The radiative correction to the electron lines with two photons exchanged to the proton may be written as $L_{\nu\sigma}$. This includes external and internal self-energies, vertex graphs, and the spanning diagram (Fig. 1). The calculation of $L_{\nu\sigma}$ proceeds in a manner similar to that of Sapirstein, Terray, and Yennie,[13] in their evaluation of radiative recoil to hyperfine structure. After mass renormalization the various radiative corrections are combined without carrying out conventional on-shell renormalization. This procedure has the advantage of allowing the ultraviolet divergences to cancel, as they must because of the Ward identity, while not introducing spurious infrared problems created by the renormalization procedure. Of course these spurious terms would ultimately cancel, but in practice the cancellation is cumbersome to achieve.

For the analysis of the radiative photon contribution to $L_{\nu\sigma}$ we use the Fried-Yennie gauge.[14] This gauge pro-

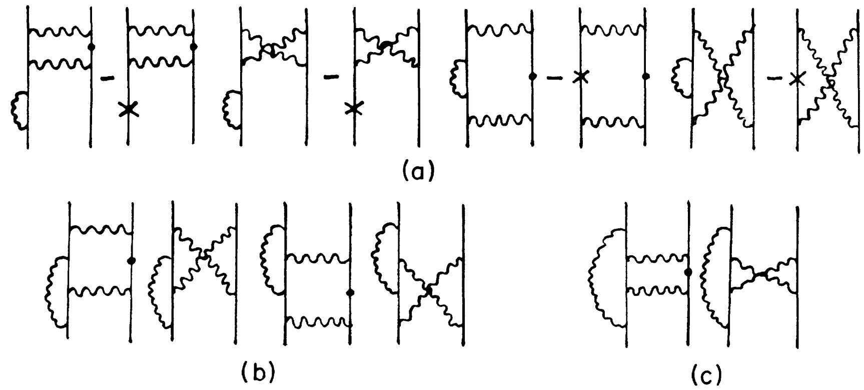

FIG. 1. Two-photon exchange perturbative kernels contributing to the radiative-recoil corrections to the Lamb shift of order $\alpha(Z\alpha)^5m^2/M$: (a) self-energy, (b) vertex, and (c) spanning photon graphs. The dot on the proton propagator in the ladder exchange graphs indicates that the on-shell positive-energy proton term has been subtracted away. The cross on the electron line in the self-energy graphs indicates the usual renormalization mass correction, δm.

vides two distinct advantages: (a) It is straightforward to prove that to order $\alpha(Z\alpha)^5m^2/M$ contributions arise only from two-photon exchange diagrams. Moreover, the external momentum can be set to zero in ΔK of Eq. (1). (b) Within this gauge, for zero external momenta, the individual graphs are free of infrared singularities, and thus we do not have to rely on delicate cancellations betweeen different graphs to obtain a finite expression.

The external-field approximation[3] was formulated in the Coulomb gauge for the exchanged photon, but the actual computation was done in a gauge-invariant manner. The correction terms we are seeking have also been shown to be gauge invariant.[15] We find that use of the Coulomb gauge for the exchanged photons provides the most transparent way to analyze additional corrections, and moreover furnishes the clearest continuity with earlier work.[3,16]

To understand this more fully we shall briefly discuss the nonradiative-recoil corrections, as given in Ref. 16, and their relation to the current work. In Ref. 16 the recoil corrections to fine structure of order $(Z\alpha)^5m^2/M$ contribute to the Lamb shift, and in the Coulomb gauge arise from (a) double Coulomb interaction, (b) single transverse photon, and (c) two transverse photons.

For the evaluation of (a) the four-dimensional exchange photon loop integration d^4p contains various poles in the variable p_0, but the *recoil contribution* comes only from an electron pole. Moreover, the loop momentum is high enough so that the external momenta can be set to zero. The resulting integration over $|\mathbf{p}|$ poses no difficulty and exhibits no infrared behavior. In the present work we calculate the radiative correction to this Coulomb-Coulomb interaction. The contribution to the recoil effect (beyond the external-field approximation) again comes only from electron poles in the variable p_0. The p_0 integral is then carried out analytically and is followed by three-dimensional numerical integration over parametric variables x, y, and the loop three-momentum $|\mathbf{p}|$.

Next consider the single transverse photon term. For nonradiative recoil this was a particularly troublesome term since an infinite number of Coulomb interactions had to be considered in the interval between emission and absorption of the transverse photon. The situation is quite different when the radiative corrections are examined. In the Fried-Yennie gauge it is established that to order $\alpha(Z\alpha)^5m^2/M$ the contributions come only from two-photon exchange and that the external momenta can be set to zero. Thus, to the desired accuracy, L_{0i} is proportional to p_i. This leads to a zero result when multiplied by the transverse photon factor $\delta_{ij}-p_ip_j/\mathbf{p}^2$.

Let us now turn to the double transverse contribution of Ref. 16. This contribution is a "seagull" diagram which is not contained in the external-field approximation. In the external-field approximation the double transverse term couples only positive-energy proton states and is of order $1/M^2$. In contrast to this the seagull is of order $1/M$ and arises from a coupling of positive-energy initial and final states to negative-energy intermediate states. The contribution of the seagull to nonradiative recoil *required* retention of external momenta in order to cut off an infrared divergence of the loop integral on $|\mathbf{p}|$. That divergence occurred when both exchanged photons became on shell at small momenta. The appearance in the result [Eq. (4.36) of Ref. 16] of $\ln\alpha$ is a remnant of that infrared behavior. When we consider the radiatively corrected seagull contribution circumstances are quite different, and the infrared problem does not occur. The reason for this is to be found in the low-energy theorem for Compton scattering. In effect, all radiative corrections to the electron line for low-frequency Compton scattering vanish since the low-frequency limit must give the Thomson result with a renormalized charge and mass. The charge renormalization occurs only through vacuum polarization effects on the photon lines. Thus the seagull graph, which is a contribution beyond the external-field approximation, and thus requires no lower order subtraction, is free of any infrared difficulties.

We find that for P states the contribution of this order is negligible and for S states we obtain

$$\Delta E_n = [\alpha(Z\alpha)^5 m^2/n^3 M] I, \qquad (2)$$

where

$$I \cong -\frac{1}{\pi^3}\mathrm{Re}\int_0^\infty dp \int_{-\infty}^{\infty} dp_4 \left[\frac{2}{3}\frac{\mathbf{p}^2 l_{ii}(ip_4,p)}{(p_4^2+\mathbf{p}^2)^2} - \frac{l_{00}(ip_4,p)-l_{00}(-\mathbf{p}^2/2M,p)}{p_4^2+(\mathbf{p}^2)^2/4M^2}\right] \qquad (3)$$

with $l_{\nu\sigma}$ the electron line radiative correction defined by

$$L_{\nu\sigma} = -\frac{i\pi^2(4\pi\alpha)^2}{m(2\pi)^4} l_{\nu\sigma}. \qquad (4)$$

Here $L_{\nu\sigma}$ is the total electron side. At this stage $l_{\nu\sigma}$ is a two-dimensional integral of an extremely complicated expression. We shall not present its explicit form in this paper but will publish it in a more complete article. We wish to remark that in the Coulomb gauge it is necessary to retain terms proportional to exchange photon momenta p_ν or p_σ since these make a nonvanishing contribution when they appear in l_{00}. In the Feynman gauge these terms would not be needed since they are gauge terms which would disappear when contracted with the proton side of the graph.

Equation (3) is the form obtained after carrying out a Wick rotation of the p_0 contour integration. After this change, p_0 was replaced by ip_4. The evaluation of I is carried out by calculating separately I_{00} and I_{ii} pieces. For I_{00}, note the presence of a subtraction which corresponds to removing from the double Coulomb piece that which is already contained in the external-field approximation.

For I_{00}, we first carry out the p_4 integral analytically by closing the contour of integration in the upper half of the complex plane. There is a pole at $p_4 = i\mathbf{p}^2/2M$ but at this pole the numerator vanishes as a consequence of the subtraction. The remaining poles are contained within $l_{00}(ip_4,p)$ and these all produce recoil corrections. In fact, at these poles we drop the denominator factor $\mathbf{p}^4/4M^2$ compared to p_4^2. There are many contributions to l_{00} and after the analytic evaluation of the p_4 integration we are left with an integral on x, y (Feynman parameters), and p. After transforming the p integration to an integral from 0 to 1 we have straightforward three-dimensional numerical integration. Our result is

$$I_{00} = -0.265 \pm 0.004, \qquad (5)$$

where the error quoted is statistical. The numerical integration was carried out by use of Monte Carlo program VEGAS.[17]

For I_{00} it was economical to carry out the p_4 integral analytically since only electron poles contributed, but we find that the I_{ii} or seagull term is easier to evaluate by direct four-dimensional integration. We transform both p_4 and p to variables which range from 0 to 1. It turns out to be extremely useful to rearrange the various terms of l_{ii} in order to avoid spurious infrared problems connected with small p_4 and p. The low-energy theorem, whose validity we have directly verified for our expression, guarantees the finiteness of the integral by ensuring that l_{ii} vanishes as p_4 and p approach zero. Nevertheless, different parts of l_{ii} may not have this property, and thus a rearrangement prior to numerical integration results in much better stability and smaller errors.

Our result is

$$I_{ii} = -0.150 \pm 0.001 \qquad (6)$$

for the seagull radiative-recoil contribution.

Combining Eqs. (2), (5), and (6) we find a total correction to the external-field approximation for S states of

$$\Delta E_n = [\alpha(Z\alpha)^5 m^2/n^3 m](-0.415 \pm 0.004). \qquad (7)$$

Combining this with our previously published result in the external-field approximation [Eq. (5.1) of Ref. 3], we obtain the complete $\alpha(Z\alpha)^5 m^2/M$ correction,

$$\Delta E_n = \frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}\left[\tfrac{35}{4}\ln 2 - 8 + \tfrac{1}{5} + \tfrac{31}{192} + (-0.415 \pm 0.004)\right].$$

For the $2S_{1/2}$-$2P_{1/2}$ Lamb splitting of hydrogen this amounts to a correction of -2.5 kHz. It should be noted that the correction to the external-field approximation is small but not negligible.

We have also studied recoil effects on vacuum polarization (VP). The reduced mass correction to the VP on a single Coulomb interaction has been given earlier[11] and is about 0.4 kHz. Other sources of such terms come from VP on a single transverse photon and from VP corrections to the nonradiative recoil discussed earlier. To analyze each of these terms we use the vacuum polarization potential as given by Brodsky and Erickson.[18] We find that such terms either contribute an additional power of α or are numerically much smaller than those which are of interest. For example, the recoil correction with two Coulomb interactions, one of which has VP, is less than 0.1 kHz and the corresponding expression with two transverse photons (seagull) is even smaller than this. Thus the largest mass correction to $\alpha(Z\alpha)^5 m$ terms involving VP is the reduced mass term above, which has been given before.

In conclusion we see that the total radiative-recoil correction to the Lamb shift is not large enough to significantly alter the status of the comparison between theory and experiment. As pointed out in the introduction there are still other unknown contributions, namely, pure recoil of order $(Z\alpha)^6 m^2/M$ and the two-loop nonrecoil binding corrections of order $\alpha^2(Z\alpha)^5 m$ which could produce sizable corrections. Finally, we note that the mass dependence of the terms we have calculated scale in such a way that we can infer the corresponding terms for muonium. Instead of -2.53 kHz for hydrogen, we obtain -22.3 kHz for muonium.

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[1]G. P. Lepage and D. R. Yennie, in *Precision Measurements and Fundamental Constants II,* edited by B. N. Taylor and W. D. Philips, U.S. National Bureau of Standards, Publication No. 617 (U.S. GPO, Washington, D.C., 1984).

[2]T. Kinoshita and J. Sapirstein, in *Proceedings of the Ninth International Conference on Atomic Physics, Seattle, Washington, 1984,* edited by R. S. Van Dyck, Jr., and E. Norval Fortson (World Scientific, Singapore, 1985).

[3]G. Bhatt and H. Grotch, Phys. Rev. A **31**, 2794 (1985).

[4]J. R. Sapirstein, Phys. Rev. Lett. **47**, 1723 (1981).

[5]P. J. Mohr, Phys. Rev. Lett. **34**, 1050 (1975).

[6]G. W. Erickson, Phys. Rev. Lett. **27**, 780 (1971).

[7]S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981), and Metrologia **22**, 9 (1986).

[8]V. G. Pal'chekov, Yu L. Sokolov, and V. P. Yakovlev, Pis'ma Zh. Eksp. Toer. Fiz. **38**, 347 (1983) [JETP Lett. **38**, 418 (1983)].

[9]G. G. Simon, Ch. Schmidt, F. Borkowski, and V. H. Walther, Nucl. Phys. **A333**, 381 (1980).

[10]D. J. Drickey and L. N. Hand, Phys. Rev. Lett. **9**, 521 (1962); L. N. Hand, D. J. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[11]G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) **35**, 271 (1965), and **35**, 447 (1965).

[12]See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer-Verlag, New York, 1976).

[13]J. R. Sapirstein, E. A. Terray, and D. R. Yennie, Phys. Rev. D **29**, 2290 (1984).

[14]H. M. Fried and D. R. Yennie, Phys. Rev. **112**, 1391 (1958).

[15]G. Bhatt, Ph.D. thesis, The Pennsylvania State University, 1986 (unpublished).

[16]H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969).

[17]G. P. Lepage, J. Comput. Phys. **27**, 192 (1978).

[18]S. J. Brodsky and G. W. Erickson, Phys. Rev. **148**, 26 (1966).