Origin of Asymptotic Freedom in Non-Abelian Field Theories

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The origin of asymptotic freedom is clarified by a simple probabilistic argument: It is much more difficult for balls to point out the same direction than sticks and disks. This fact is used to show that the effective potentials of the two-dimensional O(N)-invariant σ models (with hierarchical kinematical energy terms) are driven into the high-temperature region by the Kadanoff-Wilson block-spin transformations, if N is larger than or equal to 3.

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Though it is widely believed that non-Abelian symmetries are responsible for mass generations in two-dimensional (2D) σ models[1] and quark confinements in 4D non-Abelian lattice gauge theories,[2] their roles are still not well understood. For example, the existence of instantons[3] may be significant, but we do not know to what extent configurations of instantons saturate the functional integrals. On the other hand, asymptotic freedom is important, but the origin of this phenomenon seems to be hidden by rather complicated perturbative calculations. Therefore any discussion has been hard beyond perturbation.

In this Letter, I clarify the origin of the asymptotic freedom by an easy probabilistic argument, and I use it to show that there exist no long-range orders in 2D O(N)-invariant σ models which have hierarchical kinematical energy terms due to Dyson, Wilson, and Ma,[4-6] if (and presumably only if[7]) N is larger than or equal to 3. See also, Gawedzki and Kupiainen[8] and Bleher and Major[9] for recent study in this direction, and Ito[10] for another type of hierarchical model.

As a typical example, I consider the Gibbs measure of the 2D O(N)-invariant σ model

$$\mu(d\phi)=(Z_\Lambda)^{-1}\exp[-\tfrac{1}{2}\langle\phi,(-\Delta_\Lambda)\phi\rangle]\textstyle\prod_x g_0([\phi(x)]^2)d\phi(x), \tag{1}$$

where $\Lambda=[-L^K/2,L^K/2)^2\cap Z^2$ (L is an integer larger than or equal to 2 and K is an arbitrarily large integer) is a rectangular set of lattice points, $\phi(x)=(\phi_1(x),\dots,\phi_N(x))\in R^N$ is the spin variable at the lattice point $x\in\Lambda$,

$$\tfrac{1}{2}\langle\phi,(-\Delta_\Lambda)\phi\rangle=\tfrac{1}{2}\sum_{\substack{(x,y)\in\Lambda\\ |x-y|=1}}[\phi(x)-\phi(y)]^2 \tag{2a}$$

$$=\sum_x 2[\phi(x)]^2-\sum_{\substack{(x,y)\in\Lambda\\ |x-y|=1}}\phi(x)\cdot\phi(y) \tag{2b}$$

is the lattice Laplacean restricted to Λ, and Z_Λ is the normalization constant (partition function). Moreover $g_0(\phi^2)=\exp[-V_0(\phi^2)]$ is the single-spin distribution function and especially $g_0(\phi^2)=\delta(\phi^2-\kappa)$ for the Heisenberg models.

To see the effective interactions at long-distance scales, it is convenient to use the block-spin transformations[5,6,8]

$$\mu^{(n)}(\psi)=\int\mu^{(n-1)}(\phi)\prod_{x\in\Lambda_n}\delta(\psi(x)-(C\phi)(x))\prod_{x\in\Lambda_{n-1}}d\phi(x), \tag{3}$$

where $\Lambda_n=L^{-n}\Lambda\cap Z^2=[-L^{K-n}/2,\,L^{K-n}/2)^2\cap Z^2$,

$$\mu^{(0)}(\phi)=(Z_\Lambda)^{-1}\exp[-\tfrac{1}{2}\langle\phi,(-\Delta_\Lambda)\phi\rangle]\textstyle\prod g_0([\phi(x)]^2), \tag{4}$$

and

$$C:\{\phi(x):\ x\in\Lambda_{n-1}\}\to\{(C\phi)(x):\ x\in\Lambda_n\}$$

is defined by

$$(C\phi)(x)=L^{-2}\sum_{y\in\square(Lx)}\phi(y). \tag{5}$$

Here $\square(Lx)$ denotes the square of size $L\times L$, centered at Lx, whose sides are parallel to the axis. Consider the first step of (3) neglecting spins outside □(0). So the effective potential $V_1(\boldsymbol{\phi}^2)$ is roughly given by

$$g_1(\varphi^2)=\exp[-V_1(\varphi^2)]\cong e^{-E(\varphi^2)}P(\varphi^2), \tag{6}$$

where $P(\varphi^2)$ is the probability density for $L^{-2}\sum\boldsymbol{\phi}(x)=\boldsymbol{\phi}$ and $E(\varphi^2)$ is the most probable value for $\frac{1}{2}\langle\boldsymbol{\phi},(-\Delta_\Lambda)\boldsymbol{\phi}\rangle$ which is approximately estimated as $2L^2\kappa-2L^2\varphi^2$ by Eq. (2b). Here $\varphi=\|\boldsymbol{\phi}\|$. On the other hand,

$$P(\varphi^2)=\text{const}\int\delta(L^2\boldsymbol{\phi}-\sqrt{\kappa}\textstyle\sum_x\mathbf{s}(x))\prod_x d\mathbf{s}(x)=\text{const}\displaystyle\int_0^\infty\zeta^{N-1}d\zeta E_0(e^{iL^2\zeta\varphi s_1})E_0^{L^2}(e^{-i\sqrt{\kappa}\zeta s_1}), \tag{7}$$

where E_0 denotes the expectation value with respect to the unit sphere $S^{N-1}=\{\mathbf{s}=(s_1,\ldots,s_N)\in R^N\colon \|\mathbf{s}\|=1\}$ and $E_0(e^{ixs_1})=J_{-1+(N/2)}(x)/x^{-1+(N/2)}$. For small φ ($<L^{-\varepsilon}$, $\varepsilon>0$), the central-limit theorem implies[11] that $P(\varphi^2)\cong\text{const}\exp[-\text{const}NL^2\varphi^2/\kappa]$. However, for large φ which is important for the present purpose,[11]

$$P(\varphi^2)\simeq\text{const}\exp[\text{const}(N-1)L^2\ln(1-\varphi^2/\kappa)] \tag{8}$$

(constants >0), which thus drives $g_0(\varphi^2)=\delta(\varphi^2-\kappa)$ into a function $g_1(\varphi^2)$ which has a peak around at $\varphi^2=\kappa-\text{const}(N-1)$.

We wish to continue this process. One difficulty is that the original lattice Laplacean (2a) yields complicated nonlocal interactions, and so I use hierarchical Laplaceans due to Dyson[4] which realize the approximate renormalization recursion formulas invented by Wilson[5] and Ma[6]

$$\tfrac{1}{2}\langle\boldsymbol{\phi},(-\Delta_{\text{hcL}})\boldsymbol{\phi}\rangle=(4L^2)^{-1}\sum_{n=1}^{K+1}\sum_{x\in\Lambda_n}\sum_{y,y'\in\square(Lx)}[(C^{n-1}\boldsymbol{\phi})(y)-(C^{n-1}\boldsymbol{\phi})(y')]^2 \tag{9}$$

which should be compared with the original Laplacean Eq. (2a). In fact by the change of variables $(C^{n-1}\boldsymbol{\phi})(x)=(C^n\boldsymbol{\phi})(\tilde{x})+\mathbf{z}_{n-1}(x)$ with $(C\mathbf{z}_{n-1})(\tilde{x})=0$, where $\tilde{x}\in\Lambda_n$ and $x\in\square(L\tilde{x})$, it is easy to find

$$Z^{-1}\int\exp[i\boldsymbol{\alpha}(\boldsymbol{\phi}(x)-\boldsymbol{\phi}(y))+\tfrac{1}{2}\langle\boldsymbol{\phi},(\Delta_{\text{hcL}})\boldsymbol{\phi}\rangle]\prod d\boldsymbol{\phi}=\exp[-\text{const}\,\boldsymbol{\alpha}^2(\ln_L d(x,y)+O(1))],$$

where $d(x,y)$ is the hierarchical distance between x and y defined by $\min\{L^k\colon$ both $L^{-k}x_\mu$ and $L^{-k}y_\mu$ lie in $[m_\mu-\frac{1}{2},m_\mu+\frac{1}{2})$ for some $m_\mu\in Z$ ($\mu=1,2$) with $k=0,1,\ldots\}$. So Eq. (9) approximates Eq. (2a) with a reasonable accuracy.

In this approximation, Eq. (3) now reads

$$g_n(\varphi^2)e^{-L^2\varphi^2/4}=\frac{1}{\mathcal{N}}\int\delta(\boldsymbol{\phi}-L^{-2}\textstyle\sum_x\boldsymbol{\phi}(x))\prod_x f_{n-1}(\boldsymbol{\phi}^2(x))d\boldsymbol{\phi}(x), \tag{10a}$$

$$f_{n-1}(\varphi^2)=g_{n-1}(\varphi^2)e^{-\varphi^2/4}, \tag{10b}$$

where $x\in\square(0)$ and $\varphi=\|\boldsymbol{\phi}\|$, since

$$-\textstyle\sum_{y,y'}[\boldsymbol{\phi}(y)-\boldsymbol{\phi}(y')]^2=[\sum_y\boldsymbol{\phi}(y)]^2-L^2\sum_y\boldsymbol{\phi}^2(y).$$

This is the approximate renormalization recursion formula considered by Wilson, Ma, Gawedzki, and Kupiainen (for $L^2=2$). Note that the right-hand side (rhs) of Eq. (10a) is just the probability density for $\{\boldsymbol{\phi}(x)\colon x\in\square(0)\}$ to form the given block spin $\boldsymbol{\phi}$.

Without loss of generality, I assume that $\text{supp}g_0\subset[0,\kappa]$, $\kappa\gg1$, and then that $\text{supp}g_n\subset[0,\kappa]$ for all n. Moreover, without loss, I set $L^2=2$, and I have

$$g_n(\boldsymbol{\phi}^2)\exp(-\boldsymbol{\phi}^2/2)=\frac{1}{\mathcal{N}}\int f_{n-1}((\boldsymbol{\phi}+\mathbf{z})^2)f_{n-1}((\boldsymbol{\phi}-\mathbf{z})^2)d\mathbf{z}, \tag{10a'}$$

$$f_n(\boldsymbol{\phi}^2)=g_n(\boldsymbol{\phi}^2)\exp(-\boldsymbol{\phi}^2/4)\ (\geq0), \tag{10b'}$$

where $\mathcal{N}$ is a suitable normalization constant.

Proposition 1.—For any f_{n-1} (such that $f_{n-1}\geq0$, $\text{supp}f_{n-1}\subset[0,\kappa]$), if $N\geq3$, then $g_n(x)e^{-x/2}$ is monotone decreasing in $x\geq\kappa/4$. If $N\geq4$, then

$$-[\ln g_n(x)e^{-x/2}]'\geq\frac{1}{2x}+\frac{k}{\kappa-x} \tag{11}$$

for $x\geq\kappa/4$, where $k=(N-3)/2$.

Proof.—By the O(N) invariance, set $\boldsymbol{\phi}=(\varphi,0)$ and $\mathbf{z}=(s,\mathbf{u})\in R\times R^{N-1}$ so that the rhs of Eq. (10a') is $\text{const}\times\int f_{n-1}((\varphi+s)^2+u^2)f_{n-1}((\varphi-s)^2+u^2)u^{N-2}ds\,du$. Next insert $1=\int_0^\kappa\int_0^\kappa dp\,dq\,\delta(p-(\varphi+s)^2-u^2)\delta(q-(\varphi$

$-s)^2-u^2)$ and integrate over $s \in R$ and $u \in R_+$. Then

$$g_n(x)e^{-x/2}=\frac{1}{\mathcal{N}\sqrt{x}}\int\int dp\,dq\,\theta(\mu(p,q:x))[\mu(p,q:x)]^k f_{n-1}(p)f_{n-1}(q), \qquad (12)$$

where $x=\varphi^2$, $\theta(\mu)=1$ if $\mu\geq 0$, $\theta(\mu)=0$ if $\mu<0$, and

$$\mu(p,q:x)=(p+q)/2-x-(p-q)^2/16x. \qquad (13)$$

$\mu(p,q:x)$ is monotone decreasing in $x\geq |p-q|/4$, and then so are $[\mu(p,q:x)]^k$ and $\theta(\mu(p,q:x))$, and since $\kappa\geq|p-q|$, so is the $g_n(x)e^{-x/2}$ for $x\geq\kappa/4$. For $N\geq 4$, differentiate Eq. (12) to find

$$-[\ln g_n(x)e^{-x/2}]'=(2x)^{-1}+kB/A, \qquad (14a)$$

$$A=\int\int_D dp\,dq\,[\mu(p,q:x)]^k f_{n-1}(p)f_{n-1}(q), \qquad (14b)$$

$$B=\int\int_D dp\,dq\,[\mu(p,q:x)]^{k-1}[1-(p-q)^2/16x^2]f_{n-1}(p)f_{n-1}(q), \qquad (14c)$$

where (see Fig. 1)

$$D=\{(p,q)\in[0,\kappa]^2:\ \mu(p,q:x)\geq 0,\ q\leq p\}=\{(p,q):\ (2\sqrt{x}-\sqrt{p})^2\leq q\leq p\leq\kappa\}.$$

Then

$$B/A\geq\inf_D[1-(p-q)^2/16x^2]/\mu(p,q:x)=(\kappa-x)^{-1}\quad(p=q=\kappa).$$

Q.E.D.

This implies that the asymptotic freedom is quite probabilistic as well as geometrical[12] (i.e., it is more difficult for balls to point out the same direction than sticks and disks), and has a nonperturbative nature. This proposition can be used iteratively to show that if $N\geq 3$, $g_n(x)$ converges to $g_c(x)$: $g_c(0)=1$ and $g_c(x)=0$ for $x\neq 0$. Though this conclusion holds for all $N\geq 3$, I sketch a proof in the large-field region for $N\gg 3$. Assume that $-[\ln g_{n-1}(x)]'\geq\alpha(x)$ for $x\geq\kappa_{n-1}$ $(>\kappa/4\gg 1)$, where $\alpha(x)$ is a positive increasing function such that $\alpha(\kappa_{n-1})\geq Ck$, $C=O(1)\sim 2$ (say), $k=(N-3)/2$. Decompose D into D_1 and D_2, where $D_1=\{(p,q)\in D:\ x\leq q\leq p\leq\kappa\}$ and $D_2=\{(p,q)\in D:\ (2\sqrt{x}-\sqrt{p})^2\leq q\leq x\}$; see Fig. 1. Thus

$$-[\ln g_n(x)e^{-x/2}]'=(2x)^{-1}+k(B_1+B_2)/(A_1+A_2)\geq(2x)^{-1}+k\min\{B_i/A_i\}, \qquad (15)$$

where A_i's (respectively B_i's) are defined by replacing D by D_i's in Eq. (14b) [respectively, in Eq. (14c)]. As for B_1/A_1, set $p=x+\zeta$, $q=x+\xi$ so that

$$\mu(p,q:x)=(\zeta+\xi)/2-(\zeta-\xi)^2/16x\cong(\zeta+\xi)/2,\quad f_{n-1}(x+\zeta)\cong f_{n-1}(x)\exp[-(\alpha+\tfrac{1}{4})\zeta].$$

So

$$k\frac{B_1}{A_1}\geq\frac{2k}{k+1}(1-\varepsilon)[\alpha(x)+\tfrac{1}{4}] \qquad (16a)$$

by introduction of a new variable $t=\zeta+\xi$, where $\varepsilon=O(\alpha^{-1})$. As for B_2/A_2, set

$$p=p_0(q)+\zeta,\quad p_0(q)=(2\sqrt{x}-\sqrt{q})^2\geq x\ [0\leq\zeta\leq\kappa-(2\sqrt{x}-\sqrt{q})^2]$$

and use the trivial inequality to find $kB_2/A_2\geq\inf_q k\hat{B}_2(q)/\hat{A}_2(q)$, where

$$\hat{A}_2(q)=\int d\zeta[\mu(p_0+\zeta,q:x)]^k f_{n-1}(p_0+\zeta),\quad \hat{B}_2(q)=\int d\zeta[\mu(p_0+\zeta,q:x)]^{k-1}[1-(p_0+\zeta-q)^2/16x^2]f_{n-1}(p_0+\zeta),$$

and these functions do not depend on $f_{n-1}(q)$, $q\leq x$. An easy calculation [one may use $\mu(p_0+\zeta,q:x)\cong\zeta/2$] yields

$$kB_2/A_2\geq 2[\alpha(x)+\tfrac{1}{4}]. \qquad (16b)$$

Proposition 2.—If $N\gg 3$ and $-[\ln g_{n-1}(x)]'\geq\alpha(x)$ for $x\geq\kappa_{n-1}$ $(>\kappa/4>1)$ with $\alpha(x)$ satisfying the above assumptions,

$$-[\ln g_{n-1+l}(x)]'\geq\{[2k/(1+k)](1-\varepsilon)\}^l[\alpha(x)-\beta]+\beta\ (\uparrow\infty) \qquad (17)$$

for $x\geq\kappa_{n-1}$, where $\beta=O((k-1)^{-1})$.

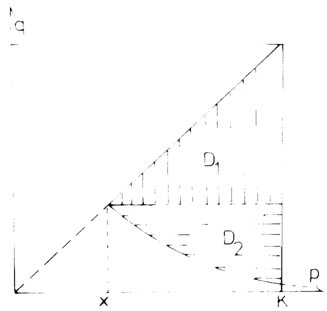

FIG. 1. Domain D and its decomposition into D_1 and D_2. By the symmetry $p \leftrightarrow q$, $0 \le q \le p$ is assumed. The parabola is $q = (2\sqrt{x} - \sqrt{p})^2$.

This proposition can be improved in the following form[13]:

Theorem 3.—(i) If $N \ge 3$, $g_n(x)$ converges to $g_c(x)$ in the sense that $-[\ln g_n(x)]' \to \infty$ as $n \to \infty$ for all x. (ii) For $N \ge 4$ (presumably for $N \ge 3$), there exist strictly positive constants $\omega > 0$ and $0 < \delta < 1$ such that $\kappa_n = \kappa - n\omega$ for $n = 1, 2, \ldots, n_0 = (\kappa - 1)/\omega$, $\kappa_n = \delta^{n-n_0}\kappa_{n_0}$ for $n \ge n_0$, and $-[\ln g_n(x)]' \ge \alpha_n(x) > 0$ for $x \ge \kappa_n$. Here $\lim \alpha_n(x) = \infty$ for all x.

Since κ_n corresponds to the effective inverse temperature at the distance scale $L^n = 2^{n/2}$, (ii) of Theorem 3 is consistent with our intuitive images for the flows of the effective interaction of the 2D Heisenberg models in low- (large-κ_n) and high- (small-κ_n) temperature regions.

The application of the present idea to the real system is under progress, in which complicated nonlocal interactions must be controlled. What is important is that the effective potentials will be forwarded into the high-temperature region (nonperturbatively) by the probabilistic driving force. Quark confinement in 4D non-Abelian lattice gauge theories[2] will be proved along this line.[13,14]

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[1]A. Polyakov, Phys. Lett. **59B**, 79 (1975).

[2]K. G. Wilson, Phys. Rev. D **10**, 2455 (1974).

[3]M. Lüscher and B. Berg, Commun. Math. Phys. **69**, 57 (1979).

[4]F. J. Dyson, Commun. Math. Phys. **21**, 269 (1971).

[5]K. Wilson and J. Kogut, Phys. Rep. **12C**, 75 (1974).

[6]S.-K. Ma, *The Modern Theory of Critical Phenomena,* (Benjamin-Cummings, Reading, MA, 1976), Chap. 8.

[7]J. F. Perez and D. H. U. Marchetti, "A Hierarchical Model Exhibiting the Kosterlitz-Thouless Fixed Points" (to be published).

[8]K. Gawedzki and A. Kupiainen, Commun. Math. Phys. **77**, 31 (1980), and **89**, 191 (1983), and **106**, 533 (1986).

[9]P. Bleher and P. Major, Commun. Math. Phys. **95**, 487 (1984).

[10]K. R. Ito, Phys. Rev. Lett. **55**, 558 (1985).

[11]F. Dunlop and C. Newman, Commun. Math. Phys. **49**, 223 (1975).

[12]There may be some relationship between this proposition and the work by E. Lieb and H. Brascamp, J. Funct. Anal. **22**, 366 (1976).

[13]K. R. Ito, to be published.

[14]Block-spin transformations for lattice gauge theories have been considered by T. Balaban, Commun. Math. Phys. **95**, 17 (1984), and **96**, 223 (1984). See also Ref. 10.