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## Nature of Spatial Chaos


P. Coullet,[a] C. Elphick, and D. Repaux

*Laboratoire de Physique Theorique, Université de Nice, 06034 Nice Cedex, France*



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We present the two basic mechanisms leading to spatial complexity in one-dimensional patterns. They are shown to be the counterparts of the horseshoe formation mechanisms studied by Melnikov and Shilnikov in dynamical systems theory.



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A lot of interest has been devoted these last few years to the phenomenon of temporal chaos,[1] which arises in spatially constrained macroscopic systems as some forcing parameter is varied. On the other hand, it is becoming increasingly popular to study the pattern formation and the transition to turbulence[2] occurring in extended systems.

The purpose of this Letter is to elucidate the nature of spatial complexity of one-dimensional patterns. A pattern will be termed spatially complex if it is described by a chaotic, but stationary in time, solution of a given partial differential equation. As in temporal systems, the mechanisms of chaos can be classified according to the mechanism of horseshoe formation.[3] We show that the two basic mechanisms are related to Melnikov's theory[3] of periodically driven one–degree-of-freedom Hamiltonian systems, and Shilnikov's theory[3] of two–degrees-of-freedom conservative systems. In both cases heteroclinic solutions of a differential equation play an important role. They are naturally associated with defects. The first mechanism has actually been identified in the context of equilibrium systems,[4,5] and is related to the pinning of defects due to an external modulation. The second one has to do with the intrinsic oscillatory nature of the defects.

In all the examples from equilibrium physics considered so far, the origin of spatial chaotic behavior has to be found in the very presence of a lattice. In macroscopic systems, the lattice, which is originally absent, is introduced via the presence of a periodic forcing. In order to illustrate the first mechanism we consider a simple model. Let $A$ be a real field obeying a Landau-Ginzburg–type equation

$$A_t = \mu A + A_{xx} - A^3 + \nu \sin(kx). \qquad (1)$$

When $\nu = 0$, Eq. (1) possesses stable localized solutions given by

$$D(x) = \pm \sqrt{\mu} \tanh[(\tfrac{1}{2}\mu)^{1/2} x]. \qquad (2)$$

For $\nu$ small enough a reasonable *Ansatz*[6] for a multidefect solution is given by

$$A(x,t) = D_i(x - x_i(t)) + \textstyle\sum_{i<j} [D_j(x - x_j(t)) - D(-\infty)] + \sum_{i>j} [D_j(x - x_j(t)) - D(\infty)], \qquad (3)$$

where $D_j$ represents the single defect centered at $x_j$. When we insert this expression in Eq. (1), the first solvability condition leads to an equation for the defect's position,

$$dx_i/d\tau = q_{i+1} E(x_{i+1} - x_i) + q_{i-1} E(x_{i-1} - x_i) + \gamma\nu \sin(kx_i), \qquad (4)$$

where $E(x_{i\pm1} - x_i) = \bar{D}(x_{i\pm1} - x_i) - D(\pm\infty)$, $\bar{D}$ is the asymptotic form of $D$, $q_j$ represents the topological charge of the $j$th defect defined as $q_j = D_j(+\infty) - D_j(-\infty)$, $\tau$ is a conveniently scaled time, and $\gamma$ is a given constant. This equation has been derived in the dilute-defects gas approximation, that is when the interdefect distances are much larger than the characteristic length of the defect cores. Equation (4) describes the dynamics of an assembly of overdamped particles on a line with nearest-neighbor interactions in an external periodic potential. Equation (1) only admits chains of alternating attractive kink and antikink whose effective interaction force reads

$$f(d)=4\mu\exp[-(2\mu)^{1/2}d], \tag{5}$$

where $d$ is the distance between two adjacent defects. When $\nu=0$, the only possible static multidefect solutions are thus found to be periodic, with an arbitrary large period. These solutions are dynamically unstable, with a characteristic evolution time $\frac{1}{2}(2\mu)^{1/2}\exp[(2\mu)^{1/2}d]$. This leads, in the case of a dilute gas, to very long transients before the system reaches its final equilibrium. Eventually, Eq. (4) no longer describes the actual dynamics, and pairs of adjacent defects are annihilated.

When $\nu\neq 0$, periodic solutions no longer have arbitrary period. They are locked with the external forcing. When the distance between defects is large enough the third term dominates. In this limit the positions of the defects are given by $x_i=2\pi n_i/k$, where the $n_i$'s are arbitrary integers. This calculation thus demonstrates the existence of chaotic behavior for Eq. (1). The stability analysis is then straightforward: Whenever an $x_i$ corresponds to a minimum of the periodic potential, the corresponding state is found to be stable, otherwise it is unstable.

An alternative way to prove the existence of spatial chaos consists of looking at Eq. (1) without its left-hand side. For $\nu=0$, the ordinary differential equation thus obtained is a one-degree-of-freedom Hamiltonian system. When $\nu\neq 0$, integrability is generically lost. More quantitatively Melnikov's analysis can be used to show explicitly the existence of horseshoes in this problem.

The other basic mechanism comes from Shilnikov's work on horseshoe formation. It is again related with the existence of a homoclinic or heteroclinic solution of a differential equation. Since we are particularly interested in defects with nonzero topological charge, we focus our attention on the heteroclinic case.[7] A heteroclinic solution of a dynamical system is a trajectory which biasymptotically connects two equilibrium solutions. For example, the defect described by Eq. (2) is a heteroclinic solution of Eq. (1) without its left-hand side, which connects the two equivalent[8] equilibrium solutions $\pm\sqrt{\mu}$. The linear behavior around these solutions dictates the asymptotic form of the heteroclinic solution. Linearization of Eq. (1) leads to opposite real eigenvalues $\alpha=\pm(2\mu)^{1/2}$. Such equilibrium solutions are termed (real) hyperbolic. Shilnikov's theory deals with homoclinic and heteroclinic solutions connecting complex hyperbolic equilibrium states. This very elegant theory allows us to demonstrate the existence of chaotic trajectories and to code them under general conditions. In the same way that Melnikov's theory could be cast in the language of defects, we now perform the same analysis in Shilnikov's case. We first give, without any particular model in mind, the equation to be satisfied by the Shilnikov defects, study it, and then consider two models of physical interest which naturally display this kind of chaotic behavior. Since now the defects present damped oscillatory tails the interacting force reads

$$f(d)=g\cos(\beta d)\exp(-\alpha d). \tag{6}$$

We consider the case where the only possible configurations consist of pairs of defects and antidefects. This situation is quite analogous to what happens in the model previously discussed. There the dynamics was attractive, because of the opposite signs of the topological charges of two nearby defects. In the presence of spatial oscillations two defects of opposite topological charges can have locally repulsive interactions. The consequence of these oscillations is to stabilize a static configuration of defects and to give rise to chaotic states. The overdamped defect's dynamics obeys a gradient-type dynamical system whose potential reads

$$\mathcal{V}[x_j]=[g/(\alpha^2+\beta^2)]\sum_i\exp[-\alpha(x_{i+1}-x_i)]\{\beta\sin[\beta(x_{i+1}-x_i)]-\alpha\cos[\beta(x_{i+1}-x_i)]\}. \tag{7}$$

A direct minimization of $\mathcal{V}$ shows that local minima exist. Actually this potential is unbounded from below, in such a way that the eventual evolution of the system cannot be captured in the defect-dynamics picture. The natural tendency of such systems is again to eliminate the defects by pair annihilation. The major difference between normal and Shilnikov defects is the possibility to construct metastable states of periodic, quasiperiodic, and even chaotic distributions of defects. Since our interest is mainly in nonequilibrium systems, such metastable configurations, in practice, can have infinite lifetime, and thus play an important role in the long-time behavior of the system. The nature of the chaotic behavior for Shilnikov-type defects is quite different from the one associated with the presence of a lattice. The oscillatory interaction between defects leads to an infinite sequence of possible equilibrium positions for the defects. Chaotic configurations arise by picking these positions at random.

We now discuss two models of physical interest displaying this kind of defect. First, an obvious generalization of Eq. (1) reads

$$A_t=\mu A+\nu A_{xx}-A_{xxxx}-A^3. \tag{8}$$

Adding the fourth-order derivative to Eq. (1) only makes sense when the coefficient in front of the diffusion term is of the order of $\sqrt{\mu}$. In this case the behavior of the system has to be represented in the two-parameter space $\mu$-$\nu$. For $\nu$ positive large enough, $\mu$ being positive, Eq. (8) can be reduced to Eq. (1). Thus in this limit, Eq.

(8) displays normal defects. Let us consider for a moment the differential equation obtained by dropping the left-hand side of Eq. (8). It is straightforward to check that, on crossing the parabola $\mu = v^2/8$, the topological nature of the equilibrium solutions $\pm\sqrt{\mu}$ changes from real hyperbolic to complex hyperbolic. Thus the defects present, inside the parabola,[9] spatial oscillations (see Fig. 1). With use of the previous technique the force between Shilnikov defects is, as anticipated, given by Eq. (6) where $g = 4\mu$, and $\alpha$ and $\beta$ respectively represent the real and imaginary parts of the corresponding eigenvalues. From a physical point of view Eq. (8) describes the dynamics associated with a so-called Lifschitz point.[10]

We now turn our attention to a model which contains richer behavior. Let $A$ be a complex field obeying the following partial differential equation:

$$A_t = (\mu - q^2)A + 2iqA_x + A_{xx} - |A|^2 A + \alpha \bar{A}^{n-1}. \tag{9}$$

This model has been used in the context of commensurate-incommensurate transitions in both equilibrium[11] and nonequilibrium[12-14] situations. In the limit $\mu \gg q^2, \alpha^{2/(4-n)}$, Eq. (9) can be reduced to a phase-type dynamics described by the overdamped sine-Gordon equation

$$\Phi_t = \Phi_{xx} - \alpha\mu^{(n-2)/2}\sin(n\Phi). \tag{10}$$

It displays kink and antikink solutions, periodic array of kink or antikink solutions, and periodic array of kink-antikink solutions [see Fig. 2(a)]. The analysis of the various solutions can be carried out in the same way as before. Out of the phase approximation the kinks present spatial oscillations, whose consequence, in particular, is to give rise to chaotic behavior [see Fig. 2(b)]. This kind of chaotic behavior is likely to be observed in experiments of the type performed by Lowe and Gollub[15] for which model (9) has been devised.[12]

We have illustrated in this Letter the two basic mechanisms for spatial complexity of one-dimensional patterns. The nature of spatial chaos is associated either with the random pinning of defects with exponential interactions by an external periodic potential or with the exponentially damped oscillatory interactions between defects, without external forcing. These mechanisms are closely related to horseshoe formations in conservative dynamical systems with one and a half[16] or two degrees of freedom. The conservative nature of these systems is a direct consequence of the parity symmetry $x \to -x$ which is not broken by a static pattern. The extension of this work to a quasi one-dimensional pattern is straightforward. It amounts to taking into account phase variations transverse to the line defect. The variational na-

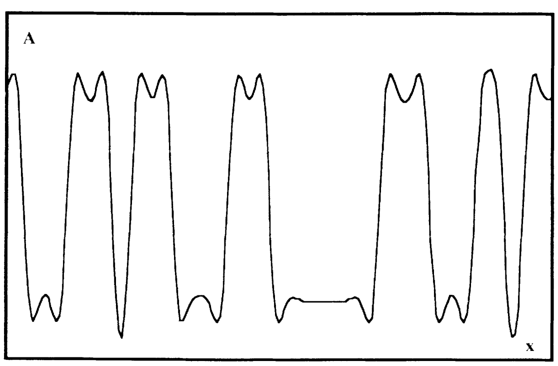


FIG. 1. A typical asymptotic solution of Eq. (8) inside the oscillatory defect's domain ($\mu = 2, v = -2$). A pseudo spectral code with 200 Fourier modes and periodic boundary conditions has been used. Length of the periodic box = 70, time step = 0.2. Initial conditions involve three Fourier modes: the fundamental, its fourth harmonic, and the seventh harmonic.

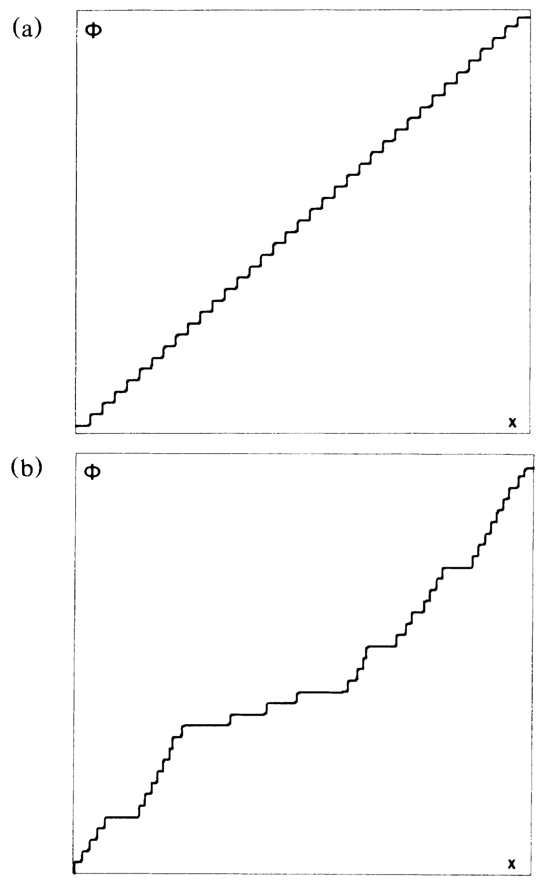


FIG. 2. Numerical simulations of the defect's dynamics Eq. (4) in the repulsive case, without external forcing ($v = 0$). The interaction force corresponds to Eq. (6) (a) for $\beta = 0$ and $\alpha = 0.3$, (b) for $\beta = 0.2$ and $\alpha = 0.3$. A fourth-order Runge-Kutta routine with the same random initial conditions has been used in both cases. The simulations involve 40 particles with periodic boundary conditions. Time step = 0.5, total integration time = 500.

ture of the models considered in this Letter is not essential. The overdamped character of the defect's dynamics is just related to the dissipative character of the basic equations describing the physical system. Conservative systems, as for example the sine-Gordon equation, would lead to inertial dynamics for the defects. For mixed systems, inertia and damping are expected. An interesting spatiotemporal complex dynamics should result from the interaction between Shilnikov's kinks. Work in these directions is in progress. The final question concerns fully developed spatiotemporal turbulence. We have described here strong spatial chaotic behaviors and hope to consider, in a future work, their weak coupling with time-dependent effects. Fully developed spatiotemporal complexity involves more complex phenomena as for example creation and annihilation of pair defects which cannot be taken into account with the methods used here.

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[a]Also at Observatoire de Nice, Nice Cedex, France.

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[7]A homoclinic orbit which biasymptotically connects the same equilibrium solution corresponds to a defect with zero topological charge. General arguments from homotopy theory show that such defects are topologically unstable.

[8]Two equilibrium solutions are equivalent if they belong to the same orbit of a discrete symmetry group which leaves the system invariant. In the cases considered here these are the reflectional symmetry and the discrete translations. When two solutions are not equivalent in this sense, the defect connecting them is generally mobile and rather called a front.

[9]The existence of kink-type solutions for Eq. (8) follows from the Hamiltonian character of this equation without its left-hand side. Heteroclinic solutions which represent these defects are shown to be persistent for this Hamiltonian system.

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[16]The terminology used comes from dynamical systems theory. One-and-a-half degree-of-freedom Hamiltonian systems are systems with one degree of freedom periodically driven by an external force.