Comment on "Transverse Electromagnetic Waves with $\mathbf{E}\parallel\mathbf{B}$"

Chu and Ohkawa[1] (CO) have proposed that a class of transverse electromagnetic (TEM) waves with $\mathbf{E}\parallel\mathbf{B}$ exists. This paper has provoked critical reaction[2-6] and a rebuttal.[7] Most of this discussion appears to be due to the failure of CO[1] to define their terminology carefully and explain their assumptions. Consequently, the respondents assumed instinctively that all TEM waves propagate and attempted to prove that $\mathbf{E}\parallel\mathbf{B}$ waves could not propagate and hence, could not exist. The example of CO[1] implied that their proposed class consisted of only standing waves and Chu[7] was explicit in his rebuttal. This Comment derives the general conditions under which TEM standing waves with $\mathbf{E}\parallel\mathbf{B}$ exist and remedies these deficiencies.

It is useful to classify TEM wave solutions of Maxwell's equations according to whether their Poynting vector $\mathbf{S}(\mathbf{r},t)=0$ or $\neq 0$. The former are $\mathbf{E}\parallel\mathbf{B}$ TEM standing waves and the latter $\mathbf{E}\perp\mathbf{B}$ TEM traveling or standing waves.[8] CO[1] made implicit use of the Coulomb gauge in their derivation. The Coulomb gauge introduces the constraints $\nabla\cdot\mathbf{A}=0$ and $\Phi(\mathbf{r},t)=C$. This gauge, which is consistent with the Lorentz gauge required to obtain independent wave equations for $\mathbf{A}$ and Φ, is used in the following analysis. However, the fields calculated are independent of the gauge.

The most general TEM solution of the vector wave equation for $\mathbf{A}$ obtained from Maxwell's equations is[9] $\mathbf{A}(\mathbf{r},t)=\mathbf{A}_+(\eta)+\mathbf{A}_-(\zeta)$, where $\eta\equiv\mathbf{k}\cdot\mathbf{r}-\omega t$ and $\zeta\equiv\mathbf{k}\cdot\mathbf{r}+\omega t$. Then $\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}=\mathbf{k}\times(\mathbf{A}'_+ +\mathbf{A}'_-)$, where $\mathbf{A}'_+\equiv[d\mathbf{A}_+(\eta)/d\eta]$ and $\mathbf{A}'_-\equiv[dA_-(\zeta)/d\zeta]$, and so $\mathbf{B}$ is transverse since $\mathbf{k}\perp\mathbf{B}$. $\nabla\cdot\mathbf{A}=\mathbf{k}\cdot(\mathbf{A}'_+ +\mathbf{A}'_-)=0$, so that $\mathbf{B}\neq 0$ if $\mathbf{A}'_+\neq -\mathbf{A}'_-$. If $\nabla\Phi=0$, then $\mathbf{E}(\mathbf{r},t)=-\partial\mathbf{A}/\partial t=\omega(\mathbf{A}'_+ -\mathbf{A}'_-)$, and so $\mathbf{E}\parallel(\mathbf{A}'_+ -\mathbf{A}'_-)$. $\mathbf{E}$ is transverse if $\mathbf{k}\cdot(\mathbf{A}'_+ -\mathbf{A}'_-)=0$. It can be shown that the Poynting vector $\mathbf{S}(\mathbf{r},t)\propto\mathbf{E}\times\mathbf{B}=\mathbf{k}[(\mathbf{A}'_+ +\mathbf{A}'_-)\cdot(\mathbf{A}'_+ -\mathbf{A}'_-)]$ for TEM waves. $\mathbf{S}(\mathbf{r},t)=0$ if $(\mathbf{A}'_+ +\mathbf{A}'_-)\cdot(\mathbf{A}'_+ -\mathbf{A}'_-)=0$, which means that $(\mathbf{A}'_+ +\mathbf{A}'_-)\perp(\mathbf{A}'_+ -\mathbf{A}'_-)$. It follows that $|\mathbf{A}'_+|=|\mathbf{A}'_-|$. If $\mathbf{A}'_+\nparallel\mathbf{A}'_-$, then $\mathbf{E}\neq 0$, $\mathbf{B}\neq 0$, and $\mathbf{E}\cdot\mathbf{B}\neq 0$. Moreover, $\mathbf{k}\cdot(\mathbf{A}'_+ +\mathbf{A}'_-)=0$ and $\mathbf{k}\cdot(\mathbf{A}'_+ -\mathbf{A}'_-)=0$ for these TEM waves. Consequently, TEM standing waves exist with $\mathbf{E}\parallel\mathbf{B}$ and $\mathbf{S}(\mathbf{r},t)=0$. The example of CO[1] satisfies these conditions.

A similar analysis for TEM traveling or standing waves with $\mathbf{E}\perp\mathbf{B}$ and $\mathbf{S}(\mathbf{r},t)\neq 0$ yields the following results: $\mathbf{E}\cdot\mathbf{B}=\omega\mathbf{k}\cdot[(\mathbf{A}'_+ +\mathbf{A}'_-)\times(\mathbf{A}'_+ -\mathbf{A}'_-)]=0$ if either $(\mathbf{A}'_+ +\mathbf{A}'_-)\parallel(\mathbf{A}'_+ -\mathbf{A}'_-)$ or $(\mathbf{A}'_+ +\mathbf{A}'_-)\times(\mathbf{A}'_+ -\mathbf{A}'_-)\perp\mathbf{k}$, and $\mathbf{S}(\mathbf{r},t)\neq 0$ if $(\mathbf{A}'_+ +\mathbf{A}'_-)\not\perp(\mathbf{A}'_+ -\mathbf{A}'_-)$.

The approach taken by CO[1] of defining another vector potential $\mathbf{F}_k(\mathbf{r})=\mathbf{A}_k(\mathbf{r})+k^{-1}\nabla\times\mathbf{A}_k(\mathbf{r})$, which leads to $\nabla\times\mathbf{F}_k(\mathbf{r})=k\mathbf{F}_k(\mathbf{r})$ if $\nabla\cdot\mathbf{F}_k(\mathbf{r})=\nabla\cdot\mathbf{A}_k(\mathbf{r})=0$, is insufficient to define those TEM standing waves with $\mathbf{E}\parallel\mathbf{B}$ and $\mathbf{S}(\mathbf{r},t)=0$, unless $\mathbf{A}_k(\mathbf{r})$ is constrained to be real since all standing waves must satisfy $\mathbf{A}(\mathbf{r},t)=\mathrm{Re}[\exp(-i\omega t)]\times\mathrm{Re}[\mathbf{A}_k(\mathbf{r})]$. Otherwise, it can be shown that the vector potential

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}_0[a\cos(\mathbf{k}\cdot\mathbf{r}+\omega t)+b\sin(\mathbf{k}\cdot\mathbf{r}+\omega t)]$$
$$=\mathrm{Re}[\exp(-i\omega t)\mathbf{A}_k(\mathbf{r})],$$

where $\mathbf{A}_k(\mathbf{r})=\mathbf{A}_0\exp[-i(\mathbf{k}\cdot\mathbf{r}-\delta)]$ and $\delta=\tan^{-1}(b/a)$, can be used to obtain a derived vector potential $\mathbf{F}_k(\mathbf{r})$ for which $\mathbf{E}\perp\mathbf{B}$ and $\mathbf{S}(\mathbf{r},t)\neq 0$.

It has been shown that a class of TEM waves with $\mathbf{E}\parallel\mathbf{B}$ exists that can be derived from a vector potential $\mathbf{A}(\mathbf{r},t)=\mathbf{A}_+(\eta)+\mathbf{A}_-(\zeta)$, satisfying $\mathbf{k}\cdot[d\mathbf{A}_+(\eta)/d\eta]=0$ and $\mathbf{k}\cdot[d\mathbf{A}_-(\zeta)/d\zeta]=0$, and a scalar potential $\Phi=C$, if $|d\mathbf{A}_+(\eta)/d\eta|=|d\mathbf{A}_-(\zeta)/d\zeta|$ and $d\mathbf{A}_+(\eta)/d\eta\parallel d\mathbf{A}_-(\zeta)/d\zeta$, where $\eta\equiv\mathbf{k}\cdot\mathbf{r}-\omega t$ and $\zeta\equiv\mathbf{k}\cdot\mathbf{r}+\omega t$. These are the most general conditions for TEM waves with $\mathbf{E}\parallel\mathbf{B}$ to exist. Those $\mathbf{E}\parallel\mathbf{B}$ solutions obtained by the condition given by CO[1] can be obtained by use of the above formalism. These waves do not propagate since $\mathbf{S}(\mathbf{r},t)=0$, and should be described as TEM standing waves with $\mathbf{E}\parallel\mathbf{B}$ to distinguish them from those classical TEM traveling and standing waves with $\mathbf{E}\perp\mathbf{B}$ and $\mathbf{S}(\mathbf{r},t)\neq 0$.

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