Resistance Noise in Nonlinear Resistor Networks

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Cohn's theorem is extended to the case of circuit elements with the nonlinear I-V characteristic $V=rI^{\alpha}$. This simplifies the study of resistance noise in nonlinear resistor networks. Exact exponent inequalities are derived. Fractal and percolating structures are considered. The infinite number of exponents necessary to characterize completely the electrical properties of linear and nonlinear percolating networks are calculated within the Migdal-Kadanoff approximation.

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Resistance noise manifests itself as voltage fluctuations when a resistor is subjected to constant current bias, or as current fluctuations in constant voltage bias. The spectrum of resistance fluctuations has, in many experimental cases of interest, a low-frequency power spectrum which varies inversely with frequency. This is the so-called $1/f$ noise. While the general origin of $1/f$ noise is not well understood, it is relatively well established that $1/f$ noise arises from resistance fluctuations.[1]

It has recently been realized that even when the microscopic origin of the resistance fluctuations is unknown, it is useful to consider the problem of the magnitude of resistance noise in disordered media.[2,3] Indeed, in such cases, one can consider the problem of finding the overall noise of the structure given the noise of the smallest elements. It was found that the magnitude of resistance noise depends on geometrical properties of the structure which are quite different from those which determine either the resistance or any of the previously defined exponents for percolation. This result is of considerable interest from both the experimental[4] and theoretical points of view. In the latter case, a whole hierarchy of exponents can be defined[2,3,5] which includes many of the known exponents describing fractals. Such families of exponents also arose in turbulence,[6] dynamical systems,[7] and diffusion-limited aggregates.[8] They are thus a general feature of fractal systems.[6]

Another recent focus of attention has then been the problem of nonlinear resistor networks. Kenkel and Straley[9,10] have considered a class of nonlinear elements which obey the current-voltage relation

$$V=r\,|I|^{\alpha}\operatorname{sgn}(I). \tag{1}$$

They have shown that this I-V characteristic is particularly interesting because it corresponds to physically realizable cases and also because (i) the I-V characteristic of an arbitrary network of such elements is also described by the same power law and (ii) close to the percolation threshold p_c, the I-V characteristic of any nonlinear network composed of monotonically increasing I-V characteristics renormalizes to such a power-law form.

In this Letter, we are interested in the problem of resistance noise in nonlinear resistance networks of the type described by Eq. (1). This problem already arose experimentally in the context of charge-density-wave systems[11] and of noise in metal-insulator mixtures.[12] It has recently been studied theoretically by Blumenfeld and Aharony.[13] Some hierarchical networks made of nonlinear elements have been considered by Arcangelis, Coniglio, and Redner,[14] who have argued that because of the symmetry of the lattices, the resistance exponents for various values of α are related to exponents for $\alpha=1$. Harris[15] has confirmed that this conjecture is valid only to first order in the nonlinearity, $\alpha-1$, for percolating systems in $d<6$.

We first present a generalization of Tellegen's and Cohn's theorems to nonlinear networks whose elements obey Eq. (1) and derive exact inequalities between exponents which are analogous to those obtained for the linear circuits.[2] Finally, the Migdal-Kadanoff (MK) position-space renormalization group allows us to obtain the infinite hierarchy of exponents for both linear and nonlinear percolating networks.

Tellegen's theorem is a consequence only of Kirchhoff's voltage and current laws and hence applies to arbitrary nonlinear networks. It can be written in the form

$$\textstyle\sum_b i_b' v_b'' = \sum_p i_p' v_p'', \tag{2}$$

where the prime and double prime refer to two circuits having the same topology, but not necessarily the same conducting elements on their bonds b or external measuring or biasing ports p. v_b'' refers to voltage drops

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across the bonds, and v_p'' refers to voltage drops at the ports but with the opposite sign convention. The proof follows exactly the lines of the linear case.[16]

Cohn's theorem[17] is the one which is particularly useful for network sensitivity analysis[16] and hence for the noise problem.[2] First note that one can take arbitrary linear combinations of Eq. (2) for different realizations of the prime circuit as defined below Eq. (2). Denoting this linear combination as $\sum_b (\Lambda' i_b) v_b'' = \sum_p (\Lambda' i_p) v_p''$, one can then apply a similar operation for the double-prime circuits to obtain the general result

$$\sum_b (\Lambda' i_b)(\Lambda'' v_b) = \sum_p (\Lambda' i_p)(\Lambda'' v_p). \tag{3}$$

We are interested in the case where the r_b are fluctuating in time. Assume that the network is polarized by a constant current I at one port and that we are measuring the fluctuating voltage V induced by the fluctuations of the overall resistance R of the network measured at that port. Let Λ' be the identity operator I, i.e., $\Lambda' i_b$ represents the current in bond b when all the elements have their time-average value r_b. Then let Λ'' be the small-increment δ operator, i.e., $\Lambda'' v_b$ is the difference between the voltage drops in this same circuit for two different realizations of the fluctuations. We restrict ourselves to the case $\alpha > 0$, where there is a uniqueness proof[10] for the solution of Kirchhoff's equations. Consider first the case where the expansion $\Lambda'' v_b \equiv \delta v_b = \delta r_b i_b^\alpha + (\partial v_b/\partial i_b)\delta i_b$ is valid. [For short, we have dropped the sgn(i) of Eq. (1).] Equation (3) then becomes

$$I\delta V = \delta R I^{\alpha+1} = \sum_b i_b \delta v_b \tag{4}$$

$$= \sum_b \delta r_b i_b^{\alpha+1} + \sum_b i_b (\partial v_b/\partial i_b)\delta i_b. \tag{5}$$

The last sum vanishes because $i_b(\partial v_b/\partial i_b)\delta i_b = \alpha \delta i_b v_b$ and $\sum_b \delta i_b v_b = \delta I V = 0$, the last equality following from Eq. (3) with $\Lambda' = \delta$, $\Lambda'' = \mathsf{I}$, and $\delta I = 0$. Note that the last term in Eq. (5) can vanish by these arguments only when $(\partial v_b/\partial i_b) i_b = C v_b$, where C is a constant. This is satisfied only for the special nonlinear I-V characteristic of interest here, so that it is only for these types of nonlinear networks that one recovers a Cohn theorem,

$$\delta R = \sum_b \delta r_b (i_b/I)^{\alpha+1}, \tag{6}$$

allowing one to compute the overall resistance fluctuations from the instantaneous values of the fluctuating resistances and the steady-state currents. In a general random-resistor network, one must allow for the fact that some of the bonds will in general carry currents so small that $i_b < \delta i_b$. In this case, the expansion of $\delta v_b - \delta r_b i_b^\alpha$ in powers of $\delta i_b / i_b$ fails (unless $\alpha = 1$). Expanding in powers of $i_b/\delta i_b$ instead, one obtains

$$\delta v_b - \delta r_b i_b^\alpha \sim r_b[\delta i_b^\alpha + \alpha i_b (\delta i_b)^{\alpha-1} - i_b^\alpha] \sim r_b \delta i_b^\alpha.$$

The contribution of these bonds to the sum in Eq. (4) is thus of order $i_b \delta i_b^\alpha < \delta i_b^{\alpha+1}$. As long as the resistance fluctuations are small enough that the δi_b are linear in the various δr, then the contribution from the bonds with $i_b < \delta i_b$ is of order $\delta i_b^{\alpha+1} \sim \delta r^{\alpha+1}$ and is thus negligible to leading order in δr_b ($\alpha > 0$). Equation (6) thus follows under rather general conditions.[18] It is also valid if δR is measured from the current fluctuations under constant voltage bias. Note that the total resistance itself obeys

$$R = \sum_b r_b (i_b/I)^{\alpha+1}, \tag{7}$$

which follows immediately from power conservation.

From Eq. (6) one can immediately derive a series of exact results using the same procedure as in Ref. 2. In particular, consider a model where each of the conducting elements of the nonlinear resistors forming the circuit has the same average value and is fluctuating independently with a correlation $\langle \delta r_b \delta r_b \rangle = \rho^2$, whose frequency dependence need not be specified. The relative noise S_R can then be calculated from

$$\mathsf{S}_R \equiv \frac{S_R}{R^2} = \frac{\langle \delta R \delta R \rangle}{R^2} = \frac{\rho^2}{r^2} \frac{\sum_b i_b^{2(\alpha+1)}}{(\sum_b i_b^{\alpha+1})^2}. \tag{8}$$

Note that the voltage noise itself would scale as $I^{2\alpha}$, instead of I^2 as in the linear case. For a homogeneous Euclidean lattice of size L, Eq. (8) predicts that $\mathsf{S}_R = (\rho^2/r^2)/L^d$. From Eq. (8), one can easily derive the composition rules for the relative noise of series and parallel resistors.

As in Refs. 2 and 5, we can, for self-similar structures, define the following infinite hierarchy of exponents x_n:

$$\sum_b (i_b/I)^{(\alpha+1)n} \approx L^{-x_n(\alpha)}. \tag{9}$$

For a nonlinear circuit with a given α, the exponents x_n, for n integer, are measurable[2] through the appropriate higher-order cumulants of the resistance fluctuations [see Eq. (6)]. Following the steps of Rammal, Tannous, and Tremblay[2] and Loève,[19] we note that $x_0 - x_n(\alpha)$ is a decreasing convex function of n, satisfying, in particular, the inequalities

$$x_{n-1}(\alpha) \le x_n(\alpha) \le \frac{n}{n-1} x_{n-1}(\alpha) - \frac{1}{n-1} x_0. \tag{10}$$

The last of these two inequalities is valid only for $n \ge 1$.

The above formalism allows us to calculate the exponents $x_n(\alpha)$ for general networks once the current distribution is known. Note also that from Eq. (9) one finds that the relation

$$x_n(\alpha) = x_{n\beta}((\alpha+1-\beta)/\beta) \tag{11}$$

is valid for structures whose currents, in any branch of the circuit, are independent of the I-V characteristic. An example of such a structure is that of Refs. 5 and 14. A special case of Eq. (11) was obtained in Ref. 14.

Let us discuss the case of percolating networks close to the threshold p_c. Let $\mathsf{S}_R \approx L^{-b}$ and $R \approx L^{-\zeta/\nu}$ be the finite-size expressions for S_R and R, respectively, which hold when the system size L is much less than the correlation length $\xi \approx (p-p_c)^{-\nu}$. Then as in Refs. 2 and 3, we have the inequalities $\tilde{\zeta}(\alpha) \le b(\alpha) \le \bar{d}_B$ where $\tilde{\zeta} \equiv \zeta/\nu$ and $\bar{d}_B$ is the fractal dimension of the conducting backbone. From the nodes-links picture, the upper bound may be improved[20] as follows: $b(\alpha) \le 2\tilde{\zeta}(\alpha) - 1/\nu$. As usual, the exponent κ defined by $S_R \approx (p-p_c)^{-\kappa}$ may be obtained from scaling, $\kappa(\alpha) = \nu[d - b(\alpha)]$, and obeys inequalities which follow from those for $b(\alpha)$. Above the upper critical dimension $d=6$, the percolating clusters have the structure of random chains so that $-x_n(\alpha) = 2$ for all values of n and α. In particular, $\tilde{\zeta}(\alpha) = b(\alpha) = \bar{d}_B = 1/\nu = \kappa(\alpha) = 2$. The conductivity exponent t, on the other hand, obeys,[9,10] for $d<6$,

$$\alpha t/\nu = \tilde{\zeta}(\alpha) + \alpha(d-1) - 1, \tag{12}$$

so that using the $d=6$ value of the right-hand side, one obtains $t/\nu = 5 + \alpha^{-1}$ for $d>6$, in agreement with the Cayley-tree results.[10]

Finally, we show that the Midgal-Kadanoff (MK) position-space renormalization group may be generalized to compute arbitrary exponents of the hierarchy for linear or nonlinear networks. Consider first the resistance of nonlinear networks. We follow closely the steps of Kirkpatrick,[21] except that we renormalize the resistance instead of the conductivity because for $\alpha=1$ this gives an exponent closer to the Monte Carlo value and because this approach can easily be generalized to arbitrary moments of the current distribution. The average resistance of blocks measuring s lattice spacings in each of the d directions is calculated by first averaging the resistance of s^{d-1} bonds occupied with probability p (bond moving) and then computing the average resistance of s bonds in series. The corresponding scaling for a homogeneous medium, $s^{1-\alpha(d-1)}$, must be divided out. Let $s = 1+\eta$ be the length rescaling, with $\eta \ll 1$. The renormalization transformation is obtained from

$$R'(\langle r\rangle) = s^{-1+\alpha(d-1)}[R_1(R_p)^{d-1}](\langle r\rangle), \tag{13}$$

where R_1 combines s resistors in series, and R_p averages s resistors in parallel. Recall the composition rules for the series, $R = \sum_i r_i$, and parallel case, $R = (\sum_i r_i^{-1/\alpha})^{-\alpha}$. R_1 yields for $\langle r\rangle$ the average resistance $\langle r'\rangle = s\langle r\rangle$ and hence in the linear approximation, $R_1(\langle r\rangle) \approx (1+\eta \times L_1)\langle r\rangle \approx (1+\eta)\langle r\rangle$. The transformation $R_p \approx 1+\eta L_p$, on the other hand, is obtained by use of the composition rule r/n^α for n parallel resistors,

$$\langle r'\rangle = \frac{1}{1-(1-p)^s} \sum_{n=1}^{s} n^{-\alpha} \frac{s!}{n!(s-n)!} p^n (1-p)^{s-n} \langle r\rangle. \tag{14}$$

The sum can be performed by first writing $n^{-\alpha} = \int_0^\infty y^{\alpha-1} e^{-ny} dy/\Gamma(\alpha)$, where Γ is Euler's gamma function. Once the sum is performed, the result can be expanded to first order in η, leading to $L_p(\langle r\rangle) = \langle r\rangle f(a,\alpha)$, where $a \equiv p_c/(1-p_c) \le 1$ and[22]

$$f(a,\alpha) = \left[\frac{\ln(1-p_c)}{p_c} + 1 - \sum_{k=1}^{\infty} \frac{-a^k}{k(k+1)^{\alpha+1}}\right]. \tag{15}$$

Substituting the values of L_1 and L_p in the first order in η expansion of Eq. (13), and recalling[10] that $\langle r\rangle \approx (p-p_c)^{-\alpha t}$, one obtains the exponent $\alpha t(\alpha)/\nu = \alpha(d-1) + (d-1)f(a,\alpha)$. From Eq. (12) one can then calculate $\tilde{\zeta}(\alpha) = -x_1(\alpha)$. The result is plotted in Fig. 1. Since, within the MK approximation, the current in each branch is independent of the I-V characteristic, Eq. (11) holds. Alternatively, one can define MK transformations directly for the various moments of the current distribution and see that Eq. (11) is satisfied. This means that with a trivial relabeling of the axes, the same curve describes the whole hierarchy of exponents, including, in particular, the noise exponent, for arbitrary values of α. We have labeled the upper horizontal axis with the value of n corresponding to $-x_n(1)$. It is clear from Fig. 1, then, that not only do the MK results have the correct convexity, they also have the correct overall shape as compared with Monte Carlo data. As expected, though, the results are better in $d=2$ than in $d=3$. One can also verify analytically that the asymptotic result[3,5,13] $-x_\infty(1) = -x_1(\infty) = 1/\nu$ is reproduced by the MK approximation, with the MK value[22] of $1/\nu$. Note,

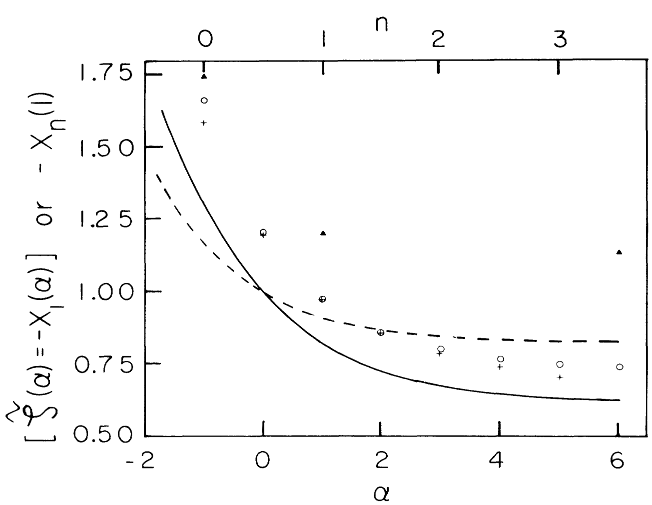

FIG. 1. Resistance exponent $\tilde{\zeta}(\alpha) = -x_1(\alpha)$ from the MK approximation: Solid line for $d=2$, and dashed line for $d=3$. From Eq. (11), these results apply, within MK, for arbitrary moments and nonlinearities $\alpha>0$. (See also Ref. 21.) In particular, $-x_1(\alpha) = -x_n(1)$ with $n = (\alpha+1)/2$. Hence, by use of the upper horizontal axis, the results can be compared to the $d=2$ Monte Carlo data adapted from Table I in Ref. 23 (crosses) and from Ref. 24 (circles). Stars are from Monte Carlo results in $d=23$ (see Ref. 20 for bibliography).

however, that while Eq. (11) trivially holds for $d > 6$, it is not expected to hold for $d < 6$, since in $d = 2$, for example, $-x_1(0) \approx 1.44$, obtained from the spreading dimension,[13] differs from $-x_{1/2}(1) \approx 1.2$, which can be read from the Monte Carlo data of Fig. 1.

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Note added.—Since this work was submitted for publication two papers on closely related subjects have appeared.[25,26]

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