Multifractal Nature of Truly Kinetic Random Walks

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We define the concept of multifractality for linear growth models. This is done on the ensemble of all trajectories and it allows us to partition this set into equivalence classes characterized by singularity strengths α and growth rates $z(\alpha)$. To extract information on the embedding properties of these subsets in the physical lattice we define a mean-square-displacement exponent $\nu(\alpha)$. The formalism is illustrated for the indefinitely growing self-avoiding walk on the square lattice with a twenty-step exact enumeration. For each of the quantities $z(\alpha)$ and $\nu(\alpha)$ we find a continuous spectrum of values.

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Through the years random walks have been studied intensively in several fields, including transport processes and polymer science. Recently it has been realized that random walks also form an important and relatively easy to study subclass of growth models.[1-4] In the more elaborate cluster growth models like diffusion-limited aggregation[5] and the stochastic model for dielectric breakdown,[6] which are known to form self-similar structures characterized by a Hausdorff dimension, it has been very recently shown that an additional continuum of scaling exponents is needed in the description of the growth-probability distribution on the interface.[7] This so called multifractal[8] or multisingular nature of fractal measures has, since the work of Halsey *et al.*,[9] gained much interest and has been studied in various physical systems, such as turbulence,[8,10] dynamical systems,[11] multiplicative random processes,[12] and localization.[13]

Ever since their introduction, the main tool in the characterization of linear aggregates grown with kinetic random-walk models has been the mean square displacement $\langle r^2\rangle_n$ of the walker after n steps and the corresponding exponent ν in $\langle r^2\rangle_n \sim n^{2\nu}$. In this Letter we show how to define the concept of multifractality for truly kinetic random walks (TKRW) like, e.g., the indefinitely growing self-avoiding walk (IGSAW).[14] From this we find that the usual exponent ν is a member of a continuous set of scaling exponents. The concepts introduced here thus provide a much more complete description of random walks than was possible previously.

The idea is to characterize each infinite-step TKRW trajectory by a scaling exponent α,[8,12] which measures how fast its total probability decays to zero with increasing step number. Because of their analogous role as in Ref. 9 we refer to the α as singularity strengths. The set of all infinite-step TKRW trajectories can then be partitioned into equivalence classes consisting of trajectories with the same singularity strength α. We then define the growth rate $z(\alpha)$ of the number of walks in these subsets and show how to extract the $z(\alpha)$ curve through a Legendre transformation on an easily computable hierarchy of exponents d_q. Although the $z(\alpha)$ and d_q are the analogs of the $f(\alpha)$ and D_q of Halsey *et al.*[9] and Hentschel and Procaccia,[15] we want to stress that the exact mapping between the two is not trivial. This becomes clear if one notes that the probability distribution used in the construction of the equivalence classes is necessarily defined on the ensemble of infinite-step TKRW trajectories,[12] whereas these subclasses for instance can be defined for one single growth cluster. As opposed to the Hausdorff dimension[9] $f(\alpha)$ of these subclasses, the $z(\alpha)$ curve contains no information on the embedding properties of the TKRW trajectories on the physical lattice. To get this information we introduce an effective mean-square-displacement exponent $\nu(\alpha)$.

The formalism will now be explained by our applying it to the IGSAW on the square lattice. The IGSAW random walker is allowed to vist with equal probability[14] any empty nearest-neighbor site which does not lead into a trap. From exact enumeration and Monte Carlo simulations[16] it is known that the correlation-length exponent is $\nu=0.567$. It has also been found[17] that the number Γ_n of n-step trajectories scales like $\Gamma_n \sim \bar{z}^n n^{\gamma-1}$, where $\bar{z}$ and γ have the self-avoiding-walk[18] values 2.64 and 1.34. Both $\bar{z}$ and ν will be shown to be members of the continuous families of exponents $z(\alpha)$ and $\nu(\alpha)$.

Let us number the n-step IGSAW trajectories with indices $i=1,\ldots,\Gamma_n$. The weight $w_n(i)$ of each n-step trajectory is then given by the product of all its one-step transition probabilities $p_i(j)$, i.e., $w_n(i)=\prod_{j=1}^n p_i(j)$. For the IGSAW these one-step transition probabilities assume the values $\frac{1}{3}$, $\frac{1}{2}$, and 1, and because it is truly kinetic it follows that $\sum_{j=1}^{\Gamma_n} w_n(j)=1$, for all n. The singularity strength α_i of the ith trajectory is defined as

$$\alpha_i = \lim_{n\to\infty} [\ln w_n(i)/\ln(\tfrac{1}{4})^n]. \tag{1}$$

The quantity $(\frac{1}{4})^n$ in the denominator is the weight of the trajectory for a symmetric random walk (SRW) on the square lattice. The reason for its appearance is that the multifractality of the IGSAW is defined with respect to its embedding in the set of all SRW trajectories. In our definition $(\frac{1}{4})^n$ plays the role of the box size[19] in Ref. 9. The α_i allow the subdivision of the set X of all infinite-step IGSAW trajectories into equivalence classes X^α of trajectories with the same value of α. For the number Γ_n^α of n-step trajectories in X^α we assume a scaling behavior similar to that for the complete set X, i.e.,

$$\Gamma_n^\alpha \sim z^n(\alpha) n^{\gamma(\alpha)-1}. \tag{2}$$

Thus the growth rate $z(\alpha)$ equals $\lim_{n\to\infty}(\Gamma_{n+1}^\alpha/\Gamma_n^\alpha)$.

The determination of the $z(\alpha)$ curve through Eqs. (1) and (2) is cumbersome. To study this curve we define a one-parameter family of exponents d_q ($q \in \mathrm{R}$) which can be more easily determined,

$$d_q = \lim_{n\to\infty} [-\ln\chi_n(q)/(q-1)\ln 4^n], \tag{3}$$

where

$$\chi_n(q) = \sum_{i=1}^{\Gamma_n} w_n^q(i), \quad q \in \mathrm{R}. \tag{4}$$

These exponents are analogs of the generalized dimensions D_q defined in Ref. 15.

From Eq. (3) it follows that $d_0=\ln\bar{z}/\ln 4=0.7$. In Ref. 19 it is shown that d_0 is the Hausdorff dimension of the IGSAW as embedded in the space of SRW trajectories. The value $d_0=0.7$ has to be compared with $d_0=1$ for the SRW itself. Now $\chi_n(q)$ is the probability for $q=2,3,\ldots$ trajectories to coincide in the first n steps. Equations (3) and (4) give

$$\lim_{n\to\infty} [\chi_{n+1}(q)/\chi_n(q)] = 4^{-(q-1)d_q}. \tag{5}$$

It thus follows that $4^{-(q-1)d_q}$ is the conditional probability for q identical n-step trajectories to proceed one step in the same direction; per trajectory this probability is therefore 4^{-d_q}. In the limit $q\to-\infty$ only those trajectories having minimal probability dominate. For the IGSAW these have $w_n=(\frac{1}{3})^n$ and one thus finds, using Eq. (3), that $d_{-\infty}=\ln 3/\ln 4=0.79$. If in the limit $q\to\infty$ the perfectly spiraling trajectories give the dominating contribution in Eq. (3), one finds $d_\infty=\ln 2/\ln 4=0.5$. However, we cannot exclude the existence of trajectories with larger probabilities. This would lead to a smaller value for d_∞. In general, 4^{-d_∞} and $4^{-d_{-\infty}}$ are the effective one-step transition probabilities for the trajectories with respect to the highest and lowest weight.

For the actual calculation of the d_q we have performed a twenty-step enumeration. In analyzing the data we assumed that the leading correction to the dominant scaling behavior of $\chi_n(q)$ in Eq. (3) is of the form n^{Δ_q}, i.e., $\chi_n(q)\sim 4^{-nd_q(q-1)}n^{\Delta_q}$. To reduce odd-even oscillations typical for the square lattice we have extrapolated d_q from

$$\frac{1}{2(q-1)}\ln\left[\frac{\chi_{n+1}(q)}{\chi_{n-1}(q)}\right] = d_q\ln 4 + \tfrac{1}{2}\Delta_q \ln\left[\frac{n+1}{n-1}\right].$$

To determine the $z(\alpha)$ curve from the d_q we first rewrite Eq. (3) using Eqs. (1) and (2) as

$$\chi_n(q) = \int d\alpha\, \rho(\alpha)(4^n)^s, \tag{6}$$
$$s \equiv -q\alpha + \ln z(\alpha)/\ln 4,$$

for large n, where we replaced the sum by an integral and introduced $\rho(\alpha)$ to include prefactors. The power $\ln z(\alpha)/\ln 4$ in the integrand of Eq. (6) can be interpreted as a Hausdorff dimension, by noting that 4^{-n} plays the role of a box size[19] as already mentioned below Eq. (1) for $\alpha(q=0)$. By performing a saddle-point approximation similar to the one in Ref. 9, one finds that $\chi_n(q)\sim(4^n)^t$, where $t\equiv -q\alpha(q)+\ln z(\alpha(q))/\ln 4$ and $\alpha(q)$ is the value of α maximizing the exponent $-\alpha q+\ln z(\alpha)/\ln 4$ in Eq. (6). From Eqs. (3) and (6) one then finds that

$$z(\alpha(q)) = 4^{q\alpha(q)-(q-1)d_q}, \tag{7a}$$

$$\alpha(q) = d[(q-1)d_q]/dq. \tag{7b}$$

By interchanging the differentiation and the limit ($n\to\infty$) in Eqs. (3) and (7b) one finds using Eq. (1) that $\alpha(q)=\lim_{n\to\infty}\sum_{i=1}^{\Gamma_n} W_n(i,q)\alpha_i$, where $W_n(i,q) = w_n^q(i)/\sum_{j=1}^{\Gamma_n} w_n^q(j)$. From this it follows that $\alpha(0) = \sum_{i=1}^{\Gamma_n}\alpha_i/\Gamma_n$ and that $\alpha(1)=d(1)=\ln z(\alpha(1))/\ln 4$. Therefore, d_1 is the Hausdorff dimension of $\chi^{\alpha(1)}$. Because $z(\alpha)$ is smaller than the coordination number of the lattice and larger than 1, we find from Eq. (7a) that $\alpha(\pm\infty)=d_{\pm\infty}$. From Eqs. (7) one can calculate the $z(\alpha)$ curve as shown in Fig. 1. In doing the saddle-point approximation it follows that $(dz/d\alpha)/z(\alpha)\ln 4=q$. So the slope at $\alpha(q=\pm\infty)$ should be infinite. The maximum should thus occur at $z(\alpha(0))=4^{d_0}=\bar{z}=2.64$, a value that we indeed find numerically (see Fig. 1). For large negative q the $z(\alpha)$ curve is almost asymptotic and one approximately finds $\alpha(-\infty)=0.79$ and $z(\alpha(-\infty)) =2.3$. The former is in good agreement with the exact value $\alpha(-\infty)=\ln 3/\ln 4$, which follows from Eq. (1) and the fact that only those trajectories with minimal weight, i.e., $\frac{1}{4},\frac{1}{3},\frac{1}{3},\ldots$, contribute. A lower bound can be derived by noting that the replacement of each bond of an arbitrary n-step trajectory by one twice its size transforms it into an element of $\Gamma_{2n}^{\alpha(-\infty)}$. Thus we find that $z^{2n}(\alpha(-\infty))\geq z^n(\alpha(q))$ for all q. Because $z(\alpha(0)) =2.64$ is the maximum one finds $z(\alpha(-\infty))\geq 1.62$. For $q\to\infty$ we expect a limiting behavior of

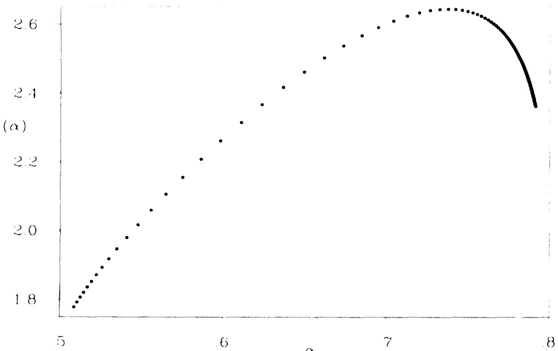

FIG. 1. The $z(\alpha)$ vs α curve for the IGSAW. The points correspond from right to left with $q=-9.0$ to 3 in steps of 0.1. The maximum occurs at $\alpha(q=0)$.

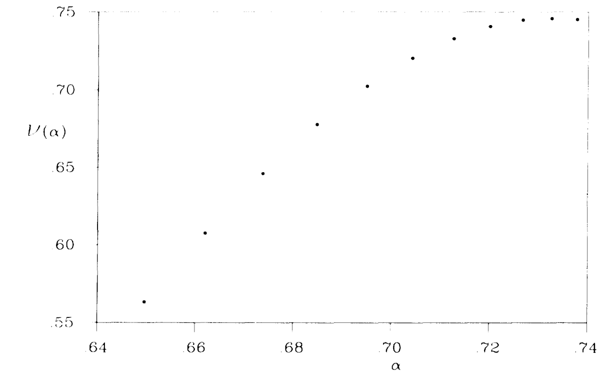

FIG. 2. The $\nu(\alpha)$ curve for the IGSAW. From right to left the α values are $\alpha(q)$, $q=0$ to 1 in steps of 0.1. The IGSAW mean-square-displacement exponent equals $\nu(\alpha(1))$.

$\alpha(q\to\infty)=0.5$, because now the dense trajectories will be selected; however, see also the discussion given for $d_{-\infty}$. The corresponding $z(\alpha(\infty))$ value is $\simeq 1.7$, but it cannot be trusted, because as can be seen from Fig. 1 no asymptotic behavior, indicated by an infinite slope, occurs for positive q values. The origin of these difficulties is obvious. For large $|q|$ and n values, w_n^q becomes very large (or small) and one is thus restricted to moderate q values. For large positive q values one has the additional problem of the lattice oscillations because only the denser IGSAW trajectories carry a substantial weight. However, for not too large values of $|q|$ we expect correct results, as indicated by the correctness of the $q=0$ result. Also for $q=1$ we find numerically the correct result $\alpha(1)=\ln z(\alpha(1))/\ln 4$.

Thus far we have studied $z(\alpha)$, a quantity which carries no information about the embedding in the square lattice. Traditionally one does not encounter this problem because the measures considered are defined on physical spaces. To get this information we study the following generalization of the mean square displacement:

$$R_n(q)=\frac{D_n(q)}{\chi_n(q)}=\sum_{i=1}^{\Gamma_n} W_n(i,q)r_i^2 \sim n^{2\nu_q}, \tag{8}$$

where $D_n(q)=\sum_{i=1}^{\Gamma_n} r_i^2 w_n^q(i)$ and r_i is the end-to-end distance of the ith n-step trajectory. We will now show that the scaling exponent ν_q in Eq. (8) is an effective mean-square-displacement exponent for the subset $X^{\alpha(q)}$. Using Eq. (1) we find

$$D_n(q)=\sum_{\alpha}(4^{-n})^{q\alpha}\sum_{i=1}^{\Gamma_n^\alpha} A_i^q r_i^2, \tag{9}$$

where the A_i are prefactors from Eq. (1). The usual scaling *Ansatz* gives

$$\sum_{i=1}^{\Gamma_n} A_i^q r_i^2 \left(\sum_{i=1}^{\Gamma_n} 1\right)^{-1} \sim n^{2\nu(\alpha)}.$$

Proceeding in the same way as in Eq. (6) we arrive at

$$D_n(q)=\int d\alpha\,\rho(\alpha)\left(4^n\right)^{s+[2\nu(\alpha)/\ln 4](\ln n)/n},$$

where s is as defined after Eq. (6). If we maximize the exponent of the integrand we obtain, up to corrections of order $n^{-1}\ln n$, the same value of $\alpha(q)$ as in Eq. (7b). Because both in the denominator and the numerator the integrands have their maximum at the same value $\alpha(q)$ we find that the $R_n(q)$ scales as $n^{2\nu(\alpha(q))}$. Note that in doing this, the correction term $n^{\gamma(\alpha)-1}$ in Eq. (2), when included in both the numerator and the denominator of Eq. (8), drops out. So in this way we have introduced a correlation-length exponent $\nu(\alpha)$ for every subset X^α, giving information about the extension in physical space for every subset separately.

In order to determine the ν_q from the series $R_n(q)$ in Eq. (8) we used the same method of analysis as in Ref. 14. From Eq. (8) it follows that $\nu(\alpha(1))$ should have the value 0.567. For $q=0$ all the trajectories have equal weight, and therefore one expects to find the self-avoiding-walk value $\frac{3}{4}$.[20] In Fig. 2 we show the $\nu(\alpha)$ curve for $\alpha=\alpha(q)$, $q=1,0.9,\ldots,0.1,0$. We find the expected value of ν at $q=0$ and $q=1$ within an accuracy of 1%. We therefore also expect the analysis to be reliable for $0<q<1$. That the $\nu(\alpha)$ is a nondecreasing function of α can be qualitatively understood by noting that the dense trajectories become more dominant for increasing q. For $q\to\infty$ one has $\nu(\alpha(\infty))=\frac{1}{2}$, but we expect that this value will be reached already for a finite positive q value. For negative q values one has $0.75\le\nu(\alpha(-\infty))\le 1$; however, we believe that $\nu(\alpha(q))$ is equal to 0.75 for all $q<0$. But to resolve this problem one needs better data.

We have shown how to define the multifractal concept for linear aggregates. Contrary to the situation in branching growth models[7] the multifractal concept developed in this paper is not a property of a single linear aggregate (trajectory) but of the complete set of possible trajectories. Although the $z(\alpha)$ curve proposed for the characterization of the multifractality can be realted to a Hausdorff dimension,[19] it does not contain information of the embedding properties in the physical lattice of the trajectories in the different subsets. To get this information we have shown how to define an effective exponent $\nu(\alpha)$ for these subsets. As an example we have applied the formalism to a simple linear growth model, the indefinetely growing self-avoiding walk. We expect that these methods are useful for the understanding of other random walks (e.g., RW's in random media,[21] Laplacian random walks[3]) as well.

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