Unique Effective Action in Five-Dimensional Kaluza-Klein Theory

S. R. Huggins,[(1)] G. Kunstatter,[(2)] H. P. Leivo,[(3)] and D. J. Toms[(1)]

[(1)]*Department of Theoretical Physics, University of Newcastle upon Tyne, Newcastle upon Tyne, United Kingdom NE1 7RU*
[(2)]*Physics Department, University of Winnipeg, Winnipeg, Manitoba, Canada R3B 2E9*
[(3)]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*

(Received 31 October 1986)

The problem of gauge and parametrization dependence in self-consistent dimensional reduction of five-dimensional quantum gravity is discussed. It is shown how the modification of the background-field method suggested by Vilkovisky is crucial for obtaining the correct result. We find that there are no physically acceptable self-consistent solutions of the form $R^4 \times S^1$ at the one-loop level.

PACS numbers: 11.10.Kk, 04.60.+n

The search for a unified field theory has led physicists to treat seriously the idea that we live in a space-time with more than four dimensions. Currently, the most promising candidate for such a theory appears to be the theory of superstrings which exists only in ten dimensions. The unobservability of the extra dimensions in more-dimensional theories is explained generally by the assumption that they are characterized by extremely small length scales. This is in fact a very old idea, the archetypal theory being that due to Kaluza[1] and Klein[2] who suggested that electromagnetism was really just a consequence of five-dimensional gravity where the fifth space-time dimension was a circle, the radius of which was on the order of the Planck length. Today nobody takes this model seriously as a possible unified field theory; however, it can still serve as a useful testing ground for some of our ideas concerning gravity in more than four dimensions. This is the spirit in which we intend the present Letter to be read.

One of the features of theories involving extra dimensions of extremely small size which manifests itself in five-dimensional gravity is the importance of quantum effects. This shows up clearly in the paper of Appelquist and Chodos[3] where they calculated the vacuum energy for pure five-dimensional gravity in the flat background $R^4 \times S^1$. One defect of this calculation is that the presence of a nonzero vacuum energy means that flat $R^4 \times S^1$ can no longer be a solution to the quantum-corrected field equations. A way around this difficulty was pointed out by Candelas and Weinberg[4] in a paper which serves as a model for most studies of quantum effects in more-dimensional theories. They pointed out that by including the cosmological constant and fixing it in an appropriate way, one could ensure that solutions which had a flat four-dimensional Minkowski space-time could be found. This procedure is usually referred to as self-consistent dimensional reduction. Candelas and Weinberg did not compute the quantum-gravity effects in such models; however, the calculations of the vacuum energy and other relevant terms have now been performed by many people.[5]

The important point which we wish to address in this Letter is that the previous calculations of quantum-gravitationally induced self-consistent dimensional reduction are incorrect. The reason for this is that the results obtained hitherto are all gauge dependent as well as dependent upon how one chooses to parametrize the theory. This shows itself in calculations of the vacuum energy performed in different gauges. The results obtained in the light-cone gauge[6] differ from those found in the Feynman gauge.[7] More recently, Kunstatter and Leivo[8] have computed the vacuum energy in a one-parameter family of covariant gauges and found a result which depends explicitly on this parameter. In addition, the calculation has also been performed for different choices of field variables: Different results were obtained.[9,10]

The two problems of the parametrization dependence and gauge dependence of the results are related. The resolution is contained in Vilkovisky's[11] approach to the effective action. Vilkovisky has pointed out that the usual formulation of the background-field method for the computation of the effective action is incorrect when applied to off-shell gravity. (More generally, the usual approach is incorrect whenever one is forced to use a connection on the space of fields which is nontrivial.) Few calculations have been done that use this new method.[12-14]

We now outline our calculations. The aim is to compute one-loop quantum effects in the effective action from the Einstein-Hilbert action

$$S[g] = -(16\pi G_0)^{-1}\int d^N x\, g^{1/2}(R - 2\Lambda_0). \tag{1}$$

In order to look for solutions of the form $R^4 \times S^1$, it is necessary and sufficient to calculate an expansion for the effective action of the form

$$\Gamma^{(1)} = -\int d^4 x\, g^{1/2}[Aa^{-4} + Ba^{-2}R + \cdots], \tag{2}$$

where A and B are calculable quantities which are detailed below, a is the radius of the extra spatial dimension, and the ellipsis indicates that higher-order curvature invariants may be dropped.

The standard approach now is to replace the metric in (1) by $g_{\mu\nu}+h_{\mu\nu}$ and expand the action to quadratic order in $h_{\mu\nu}$. This results in

$$S_2=\int d^Nx\,g^{1/2}[-\tfrac{1}{2}h^{\mu\nu}\Box h_{\mu\nu}+\tfrac{1}{4}h\Box h-(\nabla^\nu h_{\mu\nu}-\tfrac{1}{2}\nabla_\mu h)^2-h^{\mu\nu}R_{\mu\rho\nu\sigma}h^{\rho\sigma}-h^\lambda{}_\mu R^{\mu\nu}h_{\lambda\nu}+R^{\mu\nu}hh_{\mu\nu}+\tfrac{1}{2}Rh^{\mu\nu}h_{\mu\nu}-\tfrac{1}{4}Rh^2-\Lambda_0h^{\mu\nu}h_{\mu\nu}+\tfrac{1}{2}\Lambda_0h^2], \quad (3)$$

where $h=g_{\mu\nu}h^{\mu\nu}$. It is important not to take $g_{\mu\nu}=\delta_{\mu\nu}$ at this stage; otherwise it is impossible to calculate B in Eq. (2). To the classical action are then added the gauge-breaking term

$$S_{\rm GB}=\frac{1}{\alpha}\int d^Nx\,g^{1/2}(\nabla^\nu h_{\mu\nu}-\tfrac{1}{2}\nabla_\mu h)^2, \quad (4)$$

and corresponding ghost term

$$S_{\rm ghost}=\int d^Nx\,g^{1/2}V^{\mu *}[-g_{\mu\nu}\Box-R_{\mu\nu}]V^\nu, \quad (5)$$

where V^μ is a complex, anticommuting ghost field. The choice of gauge parameter $\alpha=1$ is usually made in Eq. (4) so that $S_{\rm GB}$ cancels off a similar term in S_2; however, this is not done here. The one-loop effective action may then be obtained in the usual way by our performing a functional integral.

As we have already emphasized, without implementation of Vilkovisky's ideas[11] this will lead to a gauge-parameter–dependent result. Furthermore, the result also depends on how we parametrize the fields: If instead of replacing $g_{\mu\nu}$ in Eq. (1) by $g_{\mu\nu}+h_{\mu\nu}$, we first write the higher-dimensional metric in the usual Kaluza-Klein form

$$\begin{pmatrix}\gamma_{\mu\nu}+\phi A_\mu A_\nu & \phi A_\mu\\ \phi A_\nu & \phi\end{pmatrix} \quad (6)$$

and then quantize the theory, a different result is obtained.[9,10]

The cure here is effected by our adding to the sum of Eqs. (3)–(5) the Vilkovisky correction

$$S_{\rm V}=\int d^Nx\,g^{1/2}\left[h^{\mu\lambda}R_{\mu\nu}h^\nu{}_\lambda-\tfrac{1}{2}hh^{\mu\nu}R_{\mu\nu}-\tfrac{1}{4}R(h^{\mu\nu}h_{\mu\nu}-\tfrac{1}{2}h^2)+\tfrac{1}{2}\left(\frac{N-4}{N-2}\right)\Lambda_0(h^{\mu\nu}h_{\mu\nu}-\tfrac{1}{2}h^2)\right], \quad (7)$$

with the understanding that the limit $\alpha\to 0$ is taken in the effective action. A discussion of why this is sufficient is contained in Ref. 13. It is not necessary to take $\alpha\to 0$; however, additional terms then have to be added to Eq. (7). These additional terms ensure that the final results for A and B are independent of α. Details of the calculation of A from both of these routes, and of B from the former, will be presented elsewhere.[15]

We find

$$A(\lambda)=\frac{5}{16\pi^2}d_2(-\tfrac{5}{3}\lambda)-\frac{15\zeta(5)}{64\pi^6}, \quad (8)$$

$$B(\lambda)=-\frac{13}{96\pi^2}d_1(-\tfrac{5}{3}\lambda)-\frac{3}{64\pi^2\lambda}[d_2(-\tfrac{5}{3}\lambda)-d_2(0)]-\frac{11\zeta(3)}{192\pi^4}, \quad (9)$$

where $\lambda=\Lambda_0a^2$ is dimensionless, and we have assumed $\Lambda_0<0$. The functions d_1 and d_2 are given (for $x\geq 0$) by

$$d_2(x)=-\tfrac{8}{15}\pi x^{5/2}+\sum_{n=1}^\infty\left(\frac{3}{2\pi^4n^5}+\frac{3x^{1/2}}{\pi^3n^4}+\frac{2x}{\pi^2n^3}\right)e^{-2\pi nx^{1/2}}, \quad (10)$$

$$d_1(x)=-d_2'(x). \quad (11)$$

It may be noted that the result for $A(\lambda)$ agrees with the $\alpha\to 0$ limit of the result of Ref. 8, except that the numerical coefficient of Λ_0 is different as a result of the effect of the Vilkovisky correction. This is seen directly from Eq. (7) by our imposing $R_{\mu\nu}=0$.

The field equations which must be satisfied to give $R^4\times S^1$ as a solution are easily shown to be

$$(8\pi\bar{G}_0)^{-1}\Lambda_0=a^{-4}A(\lambda), \quad (12)$$

$$2\lambda A'(\lambda)=5A(\lambda), \quad (13)$$

where $\bar{G}_0=(2\pi a)^{-1}G_0$. Equation (12) is simply the requirement that the total effective cosmological constant vanish. The true gravitational constant G is defined by the requirement that the overall coefficient of R in the effective action be $-(16\pi G)^{-1}$. This leads, upon use of the field equations, to[16]

$$a^2/G=16\pi[B(\lambda)+(2\lambda)^{-1}A(\lambda)]. \quad (14)$$

In solving for the root λ_0 of Eq. (13) one must remember that we assumed $\Lambda_0<0$, so that there is the requirement that $\lambda_0<0$. If a satisfactory root is found, Eq. (14) then gives a prediction for the radius of the extra dimension. If the right-hand side of Eq. (14) at λ_0 is positive then a is given as a multiple of the Planck length; otherwise, a is only real if the gravitational constant has the wrong sign, and hence there are no physically acceptable self-consistent solutions of the assumed form.

Solving Eq. (13) numerically gives us $\lambda_0 = -0.117615$ which satisfies the criterion that it correspond to $\Lambda_0 < 0$. Equations (8)-(11) then give $(2\lambda_0)^{-1}A(\lambda_0) = 4.09081 \times 10^{-3}$ and $B(\lambda_0) = -7.57468 \times 10^{-3}$. These values make the right-hand side of Eq. (14) negative, and hence there are no physically realistic self-consistent solutions of the desired form. Note that the induced gravity term[17] must be included here in order to obtain the correct conclusion.

In conclusion, we reiterate that Vilkovisky's[11] approach to the effective action has solved the problems of the gauge and field-parametrization dependence found in previous work on self-consistent dimensional reduction. (The problem in earlier work was that flat $R^4 \times S^1$ is not a solution to the classical field equations, and it is in this instance that Vilkovisky's approach differs from the usual one.) Although we have only demonstrated this explicitly in five-dimensional gravity, it is clear that the idea extends to more than five dimensions. As such, previous results for vacuum energies on $R^4 \times S^N$ $(N > 1)$ are incorrect.

One of us (S.R.H.) would like to thank the Science and Engineering Research Council of the United Kingdom for financial support. Another of us (G.K.) would like to thank the Natural Science and Engineering Research Council of Canada for support.

[1]T. Kaluza, Sitz. Preuss. Akad. Wiss. Berlin, Math. Phys. Kl. **1921**, 966.

[2]O. Klein, Z. Phys. **37**, 895 (1926).

[3]T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983).

[4]P. Candelas and S. Weinberg, Nucl. Phys. **B237**, 397 (1984).

[5]See D. J. Toms, Can. J. Phys. **64**, 644 (1986), for a review and the relevant references.

[6]S. Randjbar-Daemi and M. H. Sarmadi, Phys. Lett. **151B**, 343 (1985).

[7]A. Chodos and E. Myers, Ann. Phys. (N.Y.) **156**, 412 (1984); C. R. Ordóñez and M. A. Rubin, Nucl. Phys. **B260**, 456 (1985); M. H. Sarmadi, Nucl. Phys. **B263**, 187 (1986).

[8]G. Kunstatter and H. P. Leivo, Phys. Lett. **166B**, 321 (1986), and Nucl. Phys. **B279**, 641 (1987).

[9]G. Kunstatter and H. P. Leivo, Phys. Lett. B (to be published).

[10]H. P. Leivo, in "Super Field Theories," edited by H. C. Lee (Plenum, New York, to be published).

[11]G. A. Vilkovisky, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Adam Hilger, Bristol, 1984), and Nucl. Phys. **B234**, 125 (1984).

[12]A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **119**, 1 (1985).

[13]E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B234**, 509 (1984).

[14]G. Kunstatter, in Ref. 10.

[15]S. R. Huggins, G. Kunstatter, H. P. Leivo, and D. J. Toms, to be published.

[16]S. R. Huggins and D. J. Toms, Phys. Lett. **153B**, 247 (1985), and Nucl. Phys. **B263**, 433 (1986).

[17]D. J. Toms, Phys. Lett. **129B**, 31 (1983).