# Supersymmetry between Deep and Shallow Nucleus-Nucleus Potentials

D. Baye
*Physique Théorique et Mathématique, Université Libre de Bruxelles, Brussels, Belgium*
(Received 5 December 1986)

Supersymmetric quantum mechanics is applied to the determination of a shallow potential which is phase equivalent to a deep potential, and whose bound spectrum is identical except for the ground state which is suppressed. With this method, the nonphysical bound states encountered in the spectrum of microscopically founded potentials can be eliminated. The resulting shallow potentials present for small $r$ values an $r^{-2}$ singularity in accord with a generalized Levinson theorem.

PACS numbers: 24.10.Ht, 03.65.Nk, 11.30.Pb

The deep or shallow nature of nucleus-nucleus potentials has been a controversial question for a long time.[1] This problem may seem academic if both types of potentials fit accurately the same set of existing experimental data. However, their predictions differ in general in other energy domains or for properties involving an explicit use of wave functions, such as electromagnetic transition probabilities or radiative-capture cross sections.

Since experimental data do not provide a definite answer to this question, theoretical information is searched for in microscopic models, i.e., models which make use of fully antisymmetric scattering wave functions.[2] These microscopic approaches provide phase shifts which satisfy a modified Levinson theorem[3]: The forbidden states,[2] i.e., energy-independent solutions of the nonlocal microscopic equations, have to be taken into account in addition to the bound states in order to explain the phase-shift variation between zero and infinite energy. Theoretical studies based on these microscopic descriptions of scattering converge towards an accurate description of the nucleus-nucleus interaction in terms of deep local potentials.[4,5] These potentials possess a number of unphysical bound states which simulate the forbidden states of the microscopic approach. The additional bound states determine a high-energy behavior of the phase shifts which agrees with the generalized Levinson theorem. The number of forbidden states can in general be calculated in a simple way.[2] This prescription provides deep potentials which fit accurately the data for the transparent $\alpha+\alpha$ (Buck, Friedrich, and Wheatley[4]), $\alpha+{}^{16}$O (Michel *et al.*[6]), and $\alpha+{}^{40}$Ca (Michel and Vanderpoorten[7]) systems. The real part of these potentials does not depend—or depends weakly—on energy and on angular momentum.

In contrast, shallow potentials which fit the same data are often found to be strongly angular-momentum dependent, as illustrated by the $\alpha+\alpha$ example.[8] However, shallow potentials do not present the unsatisfactory occurrence of unphysical states in their bound spectra. Their bound states can be interpreted as approximations of physical states of the fused nucleus. Their main drawback is that the usual Levinson theorem prevents them from reproducing the high-energy behavior of the microscopic phase shifts. Recently, extending an idea of Swan,[9] Michel and Reidemeister[10] have shown that shallow *singular* potentials are able to reproduce the microscopic phase shifts in agreement with another extension of the Levinson theorem.[9,11] These authors derived phenomenological singular potentials which are phase equivalent to the real part of the $\alpha+{}^{16}$O deep potential.[6]

In Schrödinger quantum mechanics, Hamiltonians with identical spectra, except possibly for one bound state, can be interpreted as components of a supersymmetric Hamiltonian.[12-14] Each component can be derived from the other one with a very simple technique.[13,14] Recently, Sukumar has studied the differences between the phase shifts provided by these supersymmetric partners.[15] The aim of the present Letter is to show that supersymmetry establishes a relation between deep and shallow (but singular) local potentials. The two potentials related by this technique provide exactly the same phase shifts. Moreover, a differential operator links their wave functions. I shall rather closely follow the presentation of Sukumar,[14,15] which only involves elementary quantum mechanics. I refer the reader to Refs. 12–15 for additional information about supersymmetry.

For the $l$th partial wave, nucleus-nucleus scattering is described by the Schrödinger radial equation,

$$H_0^l\psi_0^l=(-d^2/dr^2+V_0^l)\psi_0^l=E\psi_0^l, \tag{1}$$

where $E$ is the c.m. energy and $\psi_0^l$ is $r$ times the radial wave function. Units are chosen in such a way that $\hbar^2/2m$ (where $m$ is the reduced mass) is unity. The nucleus-nucleus potential $V_0^l$ includes nuclear, Coulomb, and centrifugal terms. In order to simplify the discussion, I assume the nuclear term to be real, local, and regular. In other words, we are mainly interested in properties of the real part of traditional optical potentials. The orbital momentum superscript $l$ appearing in (1) will in general be dropped in the following.

The Hamiltonian $H_0$ can be factorized as[13,14]

$$H_0=A_0^+A_0^-+\mathcal{E}_0, \tag{2}$$

where $\mathcal{E}_0$ is the factorization energy. The first-order differential operators $A_0^+$ and $A_0^-$ are defined by[13,14]

$$A_0^- = (A_0^+)^\dagger = -d/dr + d\ln\psi_0(\mathcal{E}_0)/dr, \quad (3)$$

where $\psi_0(\mathcal{E}_0)$ is a solution of (1) with energy $\mathcal{E}_0$. Notice that $\psi_0(\mathcal{E}_0)$ must be nodeless to ensure that $A_0^+$ and $A_0^-$ be bound. The factorization energy must therefore be smaller than or equal to the ground-state energy $E_0^{(0)}$ of $H_0$. With these definitions, one can associate to $H_0$ a supersymmetric partner

$$H_1 = A_0^- A_0^+ + \mathcal{E}_0, \quad (4)$$

which will share (almost) the same spectrum. Indeed, the Hamiltonians $H_0$ and $H_1$ can be interpreted as the diagonal elements of a supersymmetric Hamiltonian. The comparison between (2) and (4) shows that the normalized eigenfunctions $\psi_0$ and $\psi_1$ of $H_0$ and $H_1$ at energy $E$ ($\geq E_0^{(0)}$) are related by

$$\psi_1(E) = (E-\mathcal{E}_0)^{-1/2} A_0^- \psi_0(E). \quad (5)$$

Any physical eigenfunction of $H_0$ corresponds to a physical eigenfunction of $H_1$ with the same eigenvalue, unless $A_0^-\psi_0(E)$ vanishes. Equation (3) shows that (5) is not valid if $E=\mathcal{E}_0=E_0^{(0)}$. Therefore, the choice of $\mathcal{E}_0=E_0^{(0)}$ makes the spectrum of $H_1$ identical to the spectrum of $H_0$ except for the ground state. Other choices of $\mathcal{E}_0$ ($<E_0^{(0)}$) lead to fully identical spectra.

Let us take $\mathcal{E}_0=E_0^{(0)}$ in order to eliminate the ground state of $H_0$. The potential corresponding to $H_1$ is given by[14]

$$V_1 = V_0 - 2d^2\ln\psi_0(E_0^{(0)})/dr^2. \quad (6)$$

For small $r$ values, the ground-state wave function $\psi_0(E_0^{(0)})$ behaves as $r^{l+1}$ so that one has[15]

$$V_1 \sim V_0 + 2(l+1)r^{-2} \sim (l+1)(l+2)r^{-2}. \quad (7)$$

The potential $V_1$ behaves as a nonsingular potential with angular momentum $l+1$. However, the supersymmetry relation only concerns the radial equation (1) of partial wave $l$: It does not modify the angular momentum from $l$ to $l+1$, as is stated in Ref. 15. The correct interpretation of (7) is that potential $V_1$ is singular. Asymptotically, the potentials $V_0$ and $V_1$ become identical since $\psi_0$ behaves as $\exp(-\gamma_0 r)$ where $\gamma_0=(-E_0^{(0)})^{1/2}$. However, in spite of this similarity, the phase shifts corresponding to the two potentials differ as shown by application in (5) of the asymptotic form $-d/dr-\gamma_0$ of $A_0^-$ [Eq. (3)] to $\sin(kr-\frac{1}{2}l\pi+\delta_0^l)$ (where $k=E^{1/2}$ and $\delta_0^l$ is the phase shift). The potentials $V_0$ and $V_1$ are not phase equivalent.

To solve this problem, I associate to $H_1$ a supersymmetric partner $H_2$ with $\mathcal{E}_1=E_0^{(0)}$ as separation energy. Notice that $\mathcal{E}_1$ is now below the ground-state energy $E_1^{(0)}$ of $H_1$. The function $\psi_1(\mathcal{E}_1)$ is therefore not square integrable and behaves asymptotically as $\exp(+\gamma_0 r)$, but can be chosen regular at $r=0$. According to the above discussion, the spectra of $H_1$ and $H_2$ are identical. To any physical state of $H_1$ with energy $E$ corresponds a state of $H_2$ with normalized wave function

$$\psi_2(E) = (E_0^{(0)}-E)^{-1} A_1^- A_0^- \psi_0(E), \quad (8)$$

where $A_1^-$ is given by (3) with $\psi_1(E_0^{(0)})$ replacing $\psi_0(\mathcal{E}_0)$. The important property of (8) is that, because of the choice of $\mathcal{E}_1$, the second-order operator $A_1^-A_0^-$ does not modify the asymptotic behavior of $\psi_0$. Hence, for any $E$, one has

$$\delta_2^l(E) = \delta_0^l(E). \quad (9)$$

The corresponding potential $V_2$ is given by

$$V_2 = V_0 - 2(d^2/dr^2)\ln[\psi_0(E_0^{(0)})\psi_1(E_0^{(0)})]. \quad (10)$$

For small $r$ values, $\psi_1(\mathcal{E}_1)$ behaves as $r^{l+2}$, as shown by (7); the behavior of $V_2$ is therefore

$$V_2 \sim V_0 + 2(2l+3)r^{-2} \sim (l+2)(l+3)r^{-2}. \quad (11)$$

Equation (9) implies that the Levinson theorem gives the same difference $\delta^l(0)-\delta^l(\infty)$ for $V_0$ and $V_2$, in spite of the fact that $V_2$ has one bound state less than $V_0$. This property is proved by an extension by Swan[11] of the Levinson theorem for potentials presenting the singularity displayed in Eq. (11).

This two-step method can be iterated until the $M$ nonphysical bound states of the deep potential are removed. The singular potential $V_{2M}$ is shallow and its bound spectrum only contains physical states. Its phase shifts $\delta_{2M}^l$ are identical to $\delta_0^l$. The chain of factorizations of supersymmetric partners provides a relation between the wave functions of $H_0$ and $H_{2M}$ as

$$\psi_{2M}(E) = (E_{2M-2}^{(0)}-E)^{-1}\cdots(E_0^{(0)}-E)^{-1} \times A_{2M-1}^- A_{2M-2}^- \cdots A_1^- A_0^- \psi_0(E). \quad (12)$$

The potential $V_{2M}$ is given by an obvious extension of (10). It behaves for small $r$ values as

$$V_{2M} \sim V_0 - 2M(2l+2M+1)r^{-2} \sim (l+2M)(l+2M+1)r^{-2}. \quad (13)$$

This behavior, which satisfies the generalized Levinson theorem,[11] has been recommended and employed by Michel and Reidemeister[10] in their phenomenological construction of shallow potentials equivalent to the deep $\alpha+{}^{16}$O potential.[6]

The chain of phase-equivalent potentials is illustrated by the $\alpha+\alpha$ scattering example in Fig. 1 (in natural units). In this case, the number of forbidden states is two for $l=0$, one for $l=2$, and zero for $l\geq 4$ (Ref. 2). The starting point is the accurate two-parameter deep potential of Buck, Friedrich, and Wheatley.[4] The $s$-wave potential (denoted as $V_0$) contains two nonphysical bound states at $-72.8$ and $-25.9$ MeV, and the $^8$Be ground state at 0.092 MeV. I employ relations (6) and

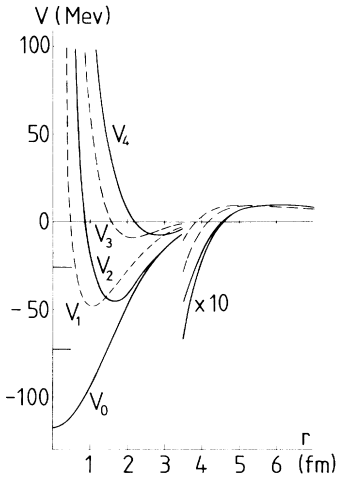


FIG. 1. Chain of $\alpha+\alpha$ potentials for $l=0$, from the deep potential $V_0$ of Buck, Friedrich, and Wheatley (Ref. 4) to the phase-equivalent shallow potential $V_4$. Potentials equivalent to $V_0$ are represented by solid lines and auxiliary potentials by dashed lines. The bound states are represented by horizontal bars.

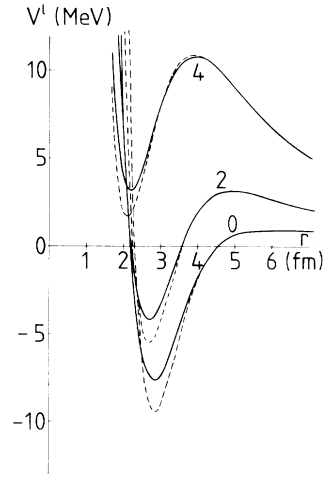


FIG. 2. Comparison between the phase-equivalent shallow potentials (solid lines) and the $\alpha+\alpha$ potentials of Ali and Bodmer (Ref. 8) (dashed lines) for $l=0,2,4$.

(10) to determine numerically the potentials $V_1$ and $V_2$. The wave functions $\psi_0(\mathcal{E}_0)$ and $\psi_1(\mathcal{E}_1)$ are computed with a standard routine of resolution of the Schrödinger equation (0.01 fm is a typical mesh size). The second-order derivatives are calculated with a five-point Lagrange derivation formula. The auxiliary potential $V_1$ (dashed line) contains only the $-25.9$-MeV bound state but provides different phase shifts. In particular, the $^8$Be ground-state resonance is not reproduced at a correct energy. Potential $V_1$ tends very slowly towards $V_0$ as $r$ becomes large. In contrast, $V_2$ has the same spectrum as $V_1$ but the same phase shifts—and resonance—as $V_0$. The phase-equivalent potential $V_2$ is very close to $V_0$ above 2.5 fm. Notice the repulsive cores of $V_1$ and $V_2$ which are due to their singular behavior [Eqs. (7) and (11)]. A second iteration eliminates the remaining bound state and provides the auxiliary potential $V_3$ (which leads to the same phase shifts as $V_1$) and the potential $V_4$ which is very similar to $V_0$ above 5 fm. Within the numerical accuracy of the computation, the phase shifts provided by $V_0$, $V_2$, and $V_4$ are identical at all energies.

The potentials $V_4^0$, $V_2^2$, and $V_0^4$ of the $l=0$, 2, and 4 partial waves are compared in Fig. 2 with the shallow $\alpha+\alpha$ potentials $d0$, $d2$, and $d4$ of Ali and Bodmer[8] (dashed lines). The similarity between the two sets of potentials is striking. The differences around the minima are most likely due to the fact that the Ali-Bodmer potentials are not singular but only possess regular repulsive cores. The Ali-Bodmer potentials can be considered as approximate supersymmetric partners of the Buck, Friedrich, and Wheatley potential.

I have also applied the present technique to the real part of the Michel *et al.* potential.[6] The resulting shallow potentials are very close to the potentials presented in Fig. 2 of Ref. 10.

In summary, supersymmetric quantum mechanics has been applied to the determination of shallow potentials which are phase equivalent to deep potentials (actually, depending on the convention chosen, the phase shifts may differ by an integer multiple of $\pi$). The shallow potentials are, however, singular and strongly $l$ dependent. This approach offers a rigorous explanation of the phenomenological shallow potentials[10] based on extensions of the Levinson theorem. The present approach is exact and provides differential relations between the wave functions of the two types of potentials. These relations should enable one to make detailed comparisons of matrix elements calculated with the two types of wave functions. The present method should be useful for the clarification of properties such as the origin of parity dependence in nucleus-nucleus scattering.[16,17]

[1]P. E. Hodgson, *Nuclear Heavy Ion Reactions* (Clarendon, Oxford, 1978).

[2]K. Wildermuth and Y. C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Branuschweig, 1977).

[3]P. Swan, Proc. Roy. Soc. London, Ser. A **228**, 10 (1955).

[4]B. Buck, H. Friedrich, and C. Wheatley, Nucl. Phys. **A275**, 246 (1977).

[5]K. Aoki and H. Horiuchi, Prog. Theor. Phys. **68**, 2028 (1982), and **69**, 1154 (1983).

[6]F. Michel, J. Albinski, P. Belery, T. Delbar, G. Gregoire, B. Tasiaux, and G. Reidemeister, Phys. Rev. C **28**, 1904 (1983).

[7]F. Michel and R. Vanderpoorten, Phys. Lett. **82B**, 183

(1979).
[8]S. Ali and A. R. Bodmer, Nucl. Phys. **80**, 99 (1966).
[9]P. Swan, Ann. Phys. (N.Y.) **48**, 455 (1968).
[10]F. Michel and G. Reidemeister, J. Phys. G **11**, 835 (1985).
[11]P. Swan, Nucl. Phys. **46**, 669 (1963).
[12]E. Witten, Nucl. Phys. **B188**, 513 (1981).
[13]A. A. Andrianov, N. V. Borisov, and M. V. Ioffe, Phys. Lett. **105A**, 19 (1984).
[14]C. V. Sukumar, J. Phys. A **18**, 2917 (1985).
[15]C. V. Sukumar, J. Phys. A **18**, 2937 (1985).
[16]D. Baye, J. Deenen, and Y. Salmon, Nucl. Phys. **A289**, 511 (1977); D. Baye, Nucl. Phys. **A460**, 581 (1986).
[17]K. Aoki and H. Horiuchi, Prog. Theor. Phys. **69**, 857 (1983).