Crossover from Linear to Nonlinear Resistance near Percolation

In a recent Letter,[1] Gefen *et al.* discussed the crossover from linear to nonlinear voltage-current characteristics of dilute resistor networks in which each individual resistor has the relation

$$V = rI + aI^{\alpha}. \tag{1}$$

Although each resistor deviates from linearity at a current $I_x^0 \sim (r/a)^{1/(\alpha-1)}$, the total network is expected to show such deviation at an external current $I_{\rm ext}^c$, which depends on both the system size L and the concentration p. Gefen *et al.* argue that $I_{\rm ext}^c \sim \Sigma_0(L)^{-y}$ for $L < \xi$ [$\Sigma_0(L)$ is the linear conductance on scale L, and $\xi \sim (p-p_c)^{-\nu}$ is the percolation connectedness length], and that $I_{\rm ext}^c \sim \Sigma^x$ for $L > \xi$. They then give bounds on x and y.

In this Comment, I give explicit values for the exponents x and y, and discuss in more detail the crossover for $L > \xi$.

Consider first the case $L < \xi$, equivalent to $p = p_c$. The total power in the network is given by

$$P = \tfrac{1}{2}\textstyle\sum_b r\,|I_b|^2 + \frac{1}{\alpha+1}\sum_b a\,|I_b|^{\alpha+1}, \tag{2}$$

where I_b is the current through the resistor on bond b of the infinite incipient cluster. The I_b's also depend implicitly on a, via the nonlinear Kirchoff equations. However, one can use a generalization of Kohn's theorem[2] to show that $(\partial P/\partial a)|_{a=0} = \sum_b |I_b^0|^{\alpha+1}/(\alpha+1)$, where I_b^0 are the currents when $a=0$. To leading (linear) order in a, we can thus replace I_b by I_b^0 in Eq. (2), and write

$$P = \tfrac{1}{2} rM(1)I^2 + [a/(\alpha+1)]M(\tfrac{1}{2}(\alpha+1))I^{\alpha+1}, \tag{3}$$

where

$$M(q) = \textstyle\sum_b (I_b^0/I)^{2q} \tag{4}$$

is the $2q$th moment of the currents in the linear problem. These moments have been studied in detail,[3] and are known to behave as $M(q) \sim L^{\tilde{\psi}(q)}$, with the multifractal exponents $\tilde{\psi}(q)$.

The linear resistance, obtained via $R = (\partial^2 P/\partial I^2)$, thus shows deviations from a constant (for $\alpha > 1$) for

$$I > I_x(L) \sim I_x^0[M(1)/M(\tfrac{1}{2}(\alpha+1))]^{1/(\alpha-1)} \sim I_x^0 L^{y\tilde{\psi}(1)} \sim \Sigma_0(L)^{-y},$$

with

$$y = [1 - \tilde{\psi}(\tfrac{1}{2}(\alpha+1))/\tilde{\psi}(1)]/(\alpha-1). \tag{5}$$

Since $\tilde{\psi}(q)$ is montonically decreasing and convex, $y(\alpha)$ is a decreasing function of α. Using estimates from Ref. 3, one has $y(\alpha) = 0.18$, 0.17, 0.08, 0 ($d=2$) and $y(\alpha) = 0.10$, 0.08, 0.06, 0 ($d=3$) for $\alpha = 0$, 1, 3, ∞ [the value at $\alpha=1$ is given by $y(1) = -\frac{1}{2}(\partial \ln\tilde{\psi}/\partial q)_{q=1}$]. $y=0$ for all α at $d > 6$. For $\alpha > 1$ we thus conclude that $0 < y < y(1)$, and the linear regime $I < I_x(L)$ extends to larger currents for larger samples. For $\alpha < 1$, the linear behavior occurs only for $I > I_x(L)$. Therefore, even a narrow nonlinear regime will be enhanced in the dilute network.

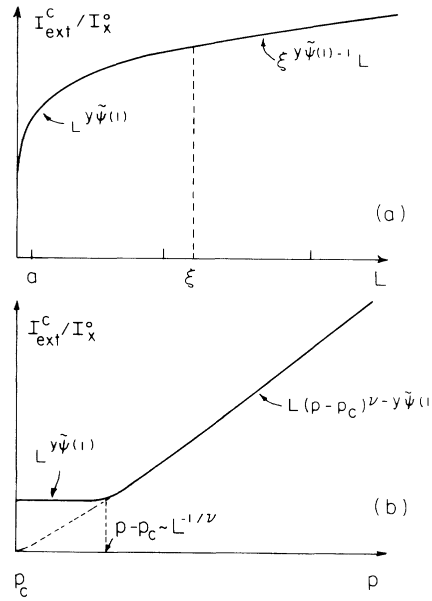

FIG. 1. Crossover current for $d=2$. (a) Fixed p and (b) fixed L.

If $L > \xi$, Gefen *et al.* write $I_{\rm ext}^c(L) = I_x(\xi)(L/\xi)^{d-1}$. Therefore, $I_{\rm ext}^c$ increases much faster with L, for fixed p, as L^{d-1} [Fig. 1(a)]. In the experimental situation described in Ref. 1, for fixed L, $I_{\rm ext}^c$ decreases as p approaches p_c from above, but reaches a plateau when ξ exceeds L [Fig. 1(b)]. Explicitly, $I_{\rm ext}^c \sim L^{d-1}[\Sigma L^{-(d-2)}]^x$, with $x = [d-1-y\tilde{\psi}(1)]/[d-2+\tilde{\psi}(1)]$. For $d=2$, this yields $x = 1.03 - y$. For $\alpha = 3$, $x = 0.97$, incompatible with the experiments of Ref. 1. Detailed experimental checks of the L dependence of $I_{\rm ext}$ on other experimental systems, and of the enhancement of the nonlinear regime when $\alpha < 1$, will be very interesting.

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