Scaling of the String Tension in a New Class of Regularized String Theories

Jan Ambjørn
The Niels Bohr Institute, 2100 Copenhagen Ø, Denmark

Bergfinnur Durhuus
Mathematics Institute, 2100 Copenhagen Ø, Denmark

and

Thordur Jonsson
NORDITA, 2100 Copenhagen Ø, Denmark
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We consider two models of discretized string theories with an action which depends on the extrinsic curvature and prove that the string tension vanishes as the coupling strength of the extrinsic curvature tends to infinity. We discuss the physical properties of the models and argue that they possess a nontrivial scaling limit.

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The bosonic string theory is presumably an ill-defined theory. This makes regularization of the world-sheet variables difficult since, by definition, the regularized string theory has nonnegative mass and hence no tachyon at the critical point, where the continuum limit is taken.

The string tension does not vanish at this critical point.[1,2] The absence of scaling of the string tension can be understood as follows: Near the critical point the dominating surfaces (at least on the lattice) contain a large number of spikes with thickness of the order of the lattice spacing. These spikes can grow and branch without any suppression since they have small area, but their entropy is so large that they alone determine the critical behavior. These surfaces have been called branched polymers. The "breather" modes, which should lower the string tension, play no role and the string tension is essentially equal to the bare string tension.[1,2] The dominance of the branched polymers and the nonscaling of the string tension is a healthy theory's version of the tachyon.

The effective bosonic string theory coming from a sensible superstring theory should not have these diseases. Hence, one would expect to be able to find a class of regularized bosonic string theories which are not dominated by branched polymers and have a string tension that scales to zero.

Apparently the only way to suppress the unwanted branched polymers effectively is by introduction of a term which shifts the critical bare string tension to zero.[2] It is not hard to see that intrinsic-curvature terms will not do the job, but it can be achieved with an extrinsic-curvature term if the coupling constant λ of that term tends to infinity as criticality is approached.

The phase diagram in the coupling-constant plane (β,λ), where β is the bare string tension, could *a priori* be as in one of the three cases shown in Fig. 1. We will show that the most natural discretized versions of string theory with extrinsic curvature[3] lead to the phase diagram of Fig. 1(b). The critical point $(\beta,\lambda)=(0,\infty)$ can be approached in such a way that the string tension and (with some assumptions) the mass scale simultaneously to zero.

We begin by discussing the influence of extrinsic curvature for the hypercubic model. The surfaces one considers in this model are made up of plaquettes from the hypercubic lattice $\mathbb{Z}^d$ glued together in such a way that one obtains a planar surface with a number of boundary components. These are the same surfaces as considered in Ref. 1 and Durhuus, Fröhlich, and Jonsson,[4] where a precise definition can be found.

The action $A(S)$ of a surface S is a special case of the action considered by Durhuus and Jonsson,[5] i.e., we define

$$A(S)=\beta\mathcal{N}(S)+\lambda\mathcal{N}'(S), \tag{1}$$

where $\mathcal{N}(S)$ is the total number of links in S and $\mathcal{N}'(S)$ is the number of links in S whose adjacent plaquettes are at right angles or overlapping. The term $\mathcal{N}'(S)$ should

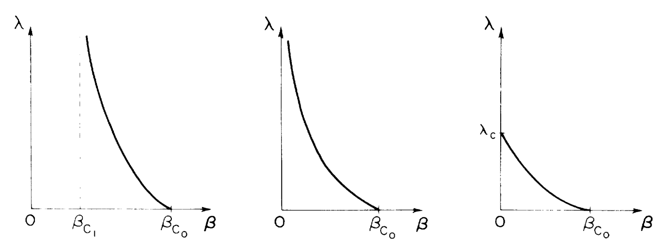

FIG. 1. The possible phase diagrams in the (β,λ) coupling-constant plane. The theory is defined on the right-hand side of the critical line.

be thought of as the extrinsic curvature of the surface S. The corresponding action for random walks is studied in Ambjørn, Durhuus, and Jonsson.[6] The loop functions are defined by

$$G_{\beta,\lambda}(\gamma_1,\ldots,\gamma_n)=\sum_{\partial S=\gamma_i\cup\cdots\cup\gamma_n} e^{-A(S)}. \qquad (2)$$

The string tension $\tau(\beta,\lambda)$ and the mass $m(\beta,\lambda)$ are defined by

$$\tau(\beta,\lambda)=-\lim_{L\to\infty} L^{-2}\ln G_{\beta,\lambda}(\gamma_{L,L}), \qquad (3)$$

$$m(\beta,\lambda)=-\lim_{a\to\infty} a^{-1}\ln G_{\beta,\lambda}(\gamma_0,\gamma_a), \qquad (4)$$

where $\gamma_{L,L}$ is a square loop with side of length L lying in a coordinate plane. γ_0 is any fixed loop, and γ_a its translate by a lattice spacings in a coordinate direction. In Refs. 1, 4, and 5 the existence and various properties of these limits are established, as well as the existence of an open convex set $\mathcal{B}$ in the (β,λ) plane such that the loop functions are finite in the interior of $\mathcal{B}$ and infinite outside the closure of $\mathcal{B}$. We shall refer to $\partial\mathcal{B}$ as the critical surface.

Our first result is that $\mathcal{B}$ has the shape indicated in Fig. 1(b), i.e., if $(\beta_c(\lambda),\lambda)\in\partial\mathcal{B}$, then $\beta_c(\lambda)\to 0$ as $\lambda\to\infty$. This leads very simply to the vanishing of τ at the critical point $\beta=0$, $\lambda=\infty$, because $G_{\beta,\lambda}(\gamma_{L,L})\geq\exp(-\beta L^2)$ and so $\tau(\beta,\lambda)\leq\beta$. The proof is based on the following estimate, whose proof is elementary, but lengthy, and will be given elsewhere.[7]

Lemma: There exists a function $f(\epsilon)$, defined for $\epsilon>0$, and positive constants c' and c'', such that

$$\mathrm{card}\{S:\ \partial S=\gamma,\ \mathcal{N}(S)=n,\ \mathcal{N}'(S)\leq\epsilon n\} = e^{f(\epsilon)n+o(n)} \qquad (5)$$

and $c'\epsilon|\ln\epsilon|\leq f(\epsilon)\leq c''\epsilon|\ln\epsilon|$ for ϵ sufficiently small.

We shall now use the lemma to show that for any given $\beta>0$ there exists $\lambda(\beta)$ such that the loop functions are finite for $\lambda>\lambda(\beta)$. It suffices to consider an arbitrary loop function corresponding to surfaces with some fixed boundary γ.

For any $\epsilon>0$ we have

$$G_{\beta,\lambda}(\gamma)=\sum_{\substack{\partial S=\gamma\\ \mathcal{N}'(S)>\epsilon\mathcal{N}(S)}} e^{-A(S)}+\sum_{\substack{\partial S=\gamma\\ \mathcal{N}'(S)\leq\epsilon\mathcal{N}(S)}} e^{-A(S)}\leq\sum_{\partial S=\gamma} e^{-(\beta+\lambda\epsilon)|S|}+\mathrm{const}\times\sum_{n=0}^{\infty} e^{-\beta n/2+f(\epsilon)n}. \qquad (6)$$

Now choose $\epsilon>0$ so small that $f(\epsilon)<\beta/2$ and take λ larger than $\epsilon^{-1}[\beta_c(0)-\beta]$. Then the right-hand side of Eq. (6) is finite and the desired result follows.

In Ref. 5 it is shown that if the susceptibility $\chi(\beta,\lambda)=\sum_{x\in\mathbb{Z}^d}G_{\beta,\lambda}(\gamma_0,\gamma_x)$ diverges as (β,λ) approaches the critical surface transversely at some point, then (with a mild extra assumption) the mass $m(\beta,\lambda)$ tends to zero. If we take this for granted, there exists, for any $\alpha>0$, a path $[0,\infty)\to\mathcal{B}$, $s\to(\beta(s),\lambda(s))$, converging to $(0,\infty)$ such that

$$\lim_{s\to\infty}\frac{m^2(\beta(s),\lambda(s))}{\tau(\beta(s),\lambda(s))}=\alpha. \qquad (7)$$

In order to prove this, take $\lambda\geq 0$ and consider the function $f_\lambda(\beta)=m^2(\beta,\lambda)/\tau(\beta,\lambda)$, which is well defined for $\beta\geq\beta_c(\lambda)$ and zero at $\beta_c(\lambda)$. It is not hard to check that $m(\beta,\lambda)\to\infty$ as $\lambda\to\infty$. Hence, for a given $\alpha>0$, there exists $\beta(\lambda)$ such that $f_\lambda(\beta(\lambda))=\alpha$. For a fixed λ the number $\beta(\lambda)$ might not be unique but we can choose the function $\lambda\to\beta(\lambda)$ to be continuous since m and τ are continuous in $\mathcal{B}$. Moreover, $\beta(\lambda)\to 0$ as $\lambda\to\infty$.

Assuming the existence of the scaling limit of the loop functions at $(\beta,\lambda)=(0,\infty)$, we obtain a continuum random surface (or string) theory with a finite renormalized mass and string tension. This theory is expected to be nontrivial because of the suppression of branched polymers.

We now turn our attention to the model of triangulated surfaces.[8-11] The model can be viewed as a regularization of Polyakov's string model, where the integration over internal metrics is replaced by a summation over random triangulations. The model has recently been shown to have a nonvanishing string tension.[2] It is, however, reassuring that the same cure as we described above for the hypercubic model also works for the triangulated one.

The model is defined by summing over all planar triangulated (piecewise linear) surfaces with a number of boundary components. The action for a fixed surface S_T, which is based on a triangulation T, can be written as

$$A(S_T)=\beta\sum_{i,j}(x_i-x_j)^2+\lambda\sum_{k,l}(1-\cos\theta_{k,l}), \qquad (8)$$

where i and j denote nearest-neighbor pairs of vertices in the triangulation T, x_i denotes the coordinate of the vertex i in the embedding space $\mathbb{R}^d$, k and l denote pairs of triangles that share an edge, and $\theta_{k,l}$ is the angle between the planes of the triangles k and l.

The second term in Eq. (8) is the most naive version of extrinsic curvature for a piecewise linear surface. It is scale invariant as is the corresponding quantity for continuum surfaces.

The loop functions are defined in analogy with Eq. (2):

$$G_{\beta,\lambda}(\gamma_1,\ldots,\gamma_r)=\sum_{\partial T=\gamma_1\cup\cdots\cup\gamma_r}\rho(T)\int dx_1\cdots dx_n\, e^{-A(S_T)}, \qquad (9)$$

where ρ is a suitable weight factor for triangulations, n is the number of internal vertices in T, and ∂T is its boundary; for more details see Ref. 8 and Ambjørn *et al.*[12] The string tension and mass are defined as in the hypercubic model by equations analogous to (3) and (4) (for details, see Ref. 2).

Our first result is a bound on the string tension:

$$\tau(\beta,\lambda) \le c_1\beta, \tag{10}$$

where c_1 is a constant. To prove (10) take a positive number κ and let T_L be a regular triangulation of order (i.e., number of internal vertices) κL^2 and with the boundary $\gamma_{L,L}$. The minimum of the action is obtained when the points x_i are regularly distributed in the plane of $\gamma_{L,L}$. If we denote this position of the vertices by x_i^0 and expand the action about this minimum, we find

$$A(S_{T_L}) \le c_1\beta L^2 + \beta\textstyle\sum_{i,j}(y_i-y_j)^2 + \lambda c_2\kappa L^2, \tag{11}$$

where $x_i = x_i^0 + y_i$ and c_2 is a constant. The term $c_1\beta L^2$ is the minimum of the action and it is easy to see that c_1 is of order 1. From Eq. (11) we obtain, by integrating, the inequality

$$G_{\beta,\lambda}(\gamma_{L,L}) \ge e^{-c_1\beta L^2 - \lambda c_2\kappa L^2}(c_3\beta^{d/2})^{-\kappa L^2}, \tag{12}$$

where c_3 is a constant of order 1. It follows that

$$\tau(\beta,\lambda) \le c_1\beta + \lambda c_2\kappa + \kappa\ln(c_3\beta^{d/2}), \tag{13}$$

from which Eq. (10) follows, since κ is arbitrary.

Further, it can be shown that for $d>2$, there exist positive constants c_4 and c_5 such that

$$c_4\beta^{-d/(d-2)} \le \lambda_c(\beta) \le c_5\beta^{-d/(d-2)}, \tag{14}$$

where $\lambda_c(\beta)$ is the critical line. The lower bound follows from a refinement of the reasoning above. To verify the upper bound one must estimate the sum over all triangulations.[7] The basic idea is to integrate successively over the vertices in a particular order and extract a factor $\beta^{-d/2}\lambda^{-(d-2)/2}$ from each integration.

The bounds (10) and (14) imply the following: (1) The scenario of Fig. 1(b) is realized for $d \ge 3$. (2) Even though the string tension is probably finite on the critical line $\lambda_c(\beta)$ for $\beta > 0$, it goes to zero as $\beta \to 0$. (3) With the assumption that the mass scales to zero for finite λ when $\beta \to \beta_c(\lambda)$, there exists a path $\beta(\lambda)$ such that Eq. (7) is satisfied. The arguments are the same as for the hypercubic model.

Finally we should mention that in $d=2$ the critical line looks as shown in Fig. 1(a). We do not know whether the string tension scales in this case.

An alternative proposal for a discretized extrinsic curvature term for randomly triangulated surfaces was suggested to us by Kazakov.[13] This term is Gaussian in the coordinates of the vertices of the surface and is given by

$$\lambda\sum_{k,l}(x_{k_{\alpha1}} - x_{k_{\alpha0}} + x_{k_{\alpha2}} - x_{l_{\alpha0}})^2, \tag{15}$$

where the summation is over all neighboring triangles k,l in a fixed triangulation and $\alpha1$, $\alpha2$ refer to vertices common to the triangles k and l while $\alpha0$ refers to the remaining two.

The phase diagram has the shape shown in Fig. 1(c). An estimate like (10) can be given for this case, and so the string tension vanishes on the line $\beta=0$, $\lambda > \lambda_c$, while it is strictly positive on the critical line from β_{c0} to λ_c.[2] The reason for the difference between this phase diagram and Fig. 1(b) is of course that the action (15) is not scale invariant.

We conclude with a few comments. It has been shown above that two different regularizations of the bosonic string lead to the same picture, i.e., the existence of a nontrivial continuum limit, where both the mass and the string tension scale.

For a fixed λ ($<\infty$) the hypercubic model belongs to the same universality class as for $\lambda=0$ under mild assumptions.[5] In particular, we know that the string tension does not scale. Only if the susceptibility does not diverge at the critical surface can we avoid that situation.[5] However, a nondiverging χ is essentially equivalent to a nonzero mass. In the hypercubic model it therefore would be very peculiar if there were a finite point (β_c,λ_c) on the critical line in Fig. 1(b) where $\tau(\beta_c,\lambda_c)=0$.

For the triangulated models it is presumably possible to extend the arguments of Ref. 2 to prove that the string tension does not vanish on the critical line.

The fixed point at which we take the continuum limit is the ultraviolet-stable fixed point $\alpha=0$ for the asymptotically free coupling constant $\alpha=\lambda^{-1}$. This is the fixed point around which the continuum string models with extrinsic curvature are expanded.[3] Polyakov[3] pointed out that there could exist another fixed point at which a theory of "smooth" surfaces could be defined. The results obtained here suggest that such a point does not exist unless one changes the model drastically, e.g., by introducing negative weight factors for surfaces as Polyakov suggested.[3]

Much remains to be done about the new class of theories discussed in this Letter. The analytical properties of the loop functions close to the critical point should be investigated and the critical exponents determined. Of extreme interest is of course the question of higher-mass excitations and, in particular, whether the mass spectrum exhibits Regge behavior.

Note added.— We have been informed by J. Fröhlich that he has obtained results similar to the ones we have found for the lattice model.

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