Logarithmic Approximations to Polynomial Lagrangeans

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A new perturbative computational scheme for solving quantum field theory is proposed. The interaction term in the Lagrangean is expanded about a free-theory form, the expansion involving powers of logarithms of the fields. The resulting perturbation series appears to have a finite radius of convergence and numerical results for simple models are good.

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Perturbation methods have played a central role in the quest for approximate solutions to quantum-field-theory models. One can distinguish between two different kinds of perturbation series: a *natural* expansion, which is a series in powers of a physical parameter that appears in the functional-integral representation of the theory, and an *artificial* expansion, which is a series in powers of a new parameter δ, which has been introduced temporarily as an expansion parameter for computational purposes. Weak-coupling expansions in powers of the coupling constant λ, strong-coupling expansions in powers of $1/\lambda$, and semiclassical (loop) expansions in powers of $\hbar$ are all natural perturbation expansions. Unfortunately these natural expansions suffer a number of disadvantages. Weak-coupling series are divergent and may not even be asymptotic to the solution of the theory. Semiclassical approximations also give divergent series, are very difficult to obtain beyond leading orders, and therefore may give very poor numerical results.[1] The computation of strong-coupling series requires the introduction of a lattice and the subsequent taking of a continuum limit; such series are often very slowly converging with many terms being required to give a reasonable approximation. The principal difficulty with natural perturbation expansions is that the analytic dependence of the solution to the theory on the physical parameters is lost; by the physical constants being forced to play the role of expansion parameters they are no longer available to display adequately the true functional dependence of the physical theory on them.[2]

The advantage of artificial perturbation expansions is that, if a parameter δ is inserted in a clever way, the resulting series in powers of δ may be easy to compute and rapidly convergent. Moreover, the terms in this expansion may exhibit a very nontrivial dependence on the physical parameters of the theory. One such perturbation scheme is the large-N expansion, where N is the number of components of a scalar field. In nonrelativistic quantum mechanics large-N expansions are surprisingly successful.[3] For a $(\mathbf{\Phi}^2)^p$ theory the very first term in the large-N expansion defines a nontrivial and renormalizable quantum field theory.[4] Also quantum chromodynamics at large N displays interesting theoretical and phenomenological features.[5]

In this Letter we propose the possibility of introducing an artificial perturbation parameter δ in the exponent of the interaction term[6]; that is, we consider a Lagrangean of the form

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\lambda M^2\phi^2(\phi^2M^{2-d})^\delta, \tag{1}$$

where d is the space-time dimensionality, λ is dimensionless, and M is a mass parameter that sets the dimensions of the interaction term. We will see that for any n, the n-point Green's function for the theory in (1) can be easily expanded as a perturbation series in powers of δ:

$$G^{(n)}(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{\infty}\delta^k g_k^{(n)}(x_1,\ldots,x_n). \tag{2}$$

We will give a simple procedure to determine the kth term in this series. We will see that in any number of space-time dimensions the terms in this series are far less divergent than the terms in the weak-coupling perturbation series. We believe that the series in (2) has a finite radius of convergence (in a future paper[7] we argue that convergence occurs for $|\delta| < 1$). With the use of Padé summation, this series even gives good numerical results for values of $|\delta| > 1$. Moreover, the Lagrangean parameters λ and μ occur in a very nontrivial way in $g_k^{(n)}$. Thus we believe that the series in (2) may reveal new information about such difficult questions in quantum field theory as whether $(\phi^4)_4$ (here $\delta=1$) is free.

It is an attractive feature of the Lagrangean L in (1) that the Green's functions have an expansion of the simple power-series form in (2). However, it is not at all obvious how to find solutions to the nonpolynomial Lagrangean that results from expanding L in (1) as a series in δ:

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2+\lambda M^2\phi^2\sum_{k=1}^{\infty}\frac{\delta^k}{k!}[\ln(\phi^2M^{2-d})]^k. \quad (3)$$

We give a combinatorial recipe for obtaining the series in number of (2) in any dimensions of space-time: The coefficient $g_0^{(n)}(x_1,\ldots,x_n)$ of δ^0 in (2) is merely the n-point Green's function for a *free* theory in which the mass term is $\mu^2+2\lambda M^2$. [This is evident from (3).] To compute the coefficients of δ^1 through δ^K, we consider a new polynomial Lagrangean $\tilde{L}$:

$$\tilde{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2+\lambda M^d\sum_{k=1}^{K}(\phi^2M^{2-d})^{\alpha_k+1}P_k. \quad (4)$$

Here, $\alpha_1,\ldots,\alpha_K$ are initially regarded as integers; P_k are polynomials in $\alpha_1,\ldots,\alpha_K$, they are polynomials of degree K in the perturbation parameter δ, and they have at least one power of δ (see below). In this new theory we must compute (using ordinary weak-coupling diagrammatic techniques) the n-point Green's function $\tilde{G}^{(n)}(x_1,\ldots,x_n)$ to order δ^K. Then we apply the derivative operator D given by

$$D=\frac{1}{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\frac{\exp[2\pi i(k-1)j/K]}{j!}\left(\frac{\partial}{\partial\alpha_k}\right)^j \quad (5)$$

to $\tilde{G}^{(n)}(x_1,\ldots,x_K)$ and evaluate the result at $\alpha_1=\alpha_2=\cdots=\alpha_K=0$. In this differentiation we can no longer consider the parameters α_i as integers; thus, the series in (2) is generated from derivatives of the theory specified by $\tilde{L}$ in (4) at the point where this theory is free. Note that, in effect, this recipe reduces the problem of finding the nonperturbative series in (2) to that of solving the polynomial Lagrangean $\tilde{L}$ by standard weak-coupling diagrammatic techniques. Moreover, only a finite number of graphs are required: To obtain the series in (2) to order δ^K, only graphs having up to K vertices are required.

It remains only to specify the polynomials P_k in (4). The first few polynomials are given below. (The procedure for finding P_k is given in Ref. 7.) For $K=1$, we have

$$P_1=\delta; \quad (6)$$

for $K=2$,

$$P_1=\delta+\delta^2,\quad P_2=-\delta+\delta^2; \quad (7)$$

for $K=3$,

$$P_1=\delta+\tfrac{1}{2}(1+\alpha_1)\delta^2+\delta^3,$$
$$P_2=\delta\omega+\tfrac{1}{2}(\omega^2+\alpha_2)\delta^2+\delta^3, \quad (8)$$
$$P_3=\delta\omega^2+\tfrac{1}{2}(\omega+\alpha_3)\delta^2+\delta^3,$$

where $\omega=e^{2\pi i/3}$.

To illustrate the perturbative computational procedure outlined in this Letter, we examine three cases: $d=0$, $d=1$ (quantum mechanics), and the large-N limit.

Case d=0.—Consider the zero-dimensional field theory whose vacuum-vacuum amplitude Z is given by

$$Z=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-(x^2)^{1+\delta}}. \quad (9)$$

The integral in (9) evaluates exactly to

$$Z=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3+2\delta}{2+2\delta}\right). \quad (10)$$

Thus the free energy $E(\delta)=-W=-\ln Z$ is clearly an analytic function of complex δ for $|\delta|<1$. (The singularities in the δ plane lie on the negative-δ axis between $\delta=-3/2$ and $\delta=-1$, with an accumulation point at $\delta=-1$.) The expansion of $E(\delta)$ directly as a series in powers of δ gives

$$E(\delta)=\tfrac{1}{2}\delta\psi(\tfrac{3}{2})-\delta^2[\tfrac{1}{2}\psi(\tfrac{3}{2})+\tfrac{1}{8}\psi'(\tfrac{3}{2})]+\cdots, \quad (11)$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$.

This power series is rapidly convergent for $|\delta|<1$. However, for all values of δ outside the range $-3/2<\delta<-1$, $|\delta|\geq 1$, superb numerical accuracy is obtained if the series in (11) is converted to a Padé approximant: The exact value of $E(1)=-0.0225104$. A (3,2) Padé approximant gives -0.02252 and a (5,4) Padé approximant gives -0.0225103.

To compute the terms in the series (11) with use of the recipe given in (4)–(8) is straightforward. We merely sum the connected vacuum bubble graphs corresponding to the Lagrangean $\tilde{L}$ in (4) and apply the operator D in (5). For example, to order δ^2 we have a theory with two vertices. The Feynman rules are (i) $\frac{1}{2}$ for a line; (ii) $-(\delta+\delta^2)(2\alpha_1)!$ for a one-vertex; (iii) $(\delta-\delta^2)(2\alpha_2)!$ for a two-vertex. The graphs that contribute to $E(\delta)$ are shown in Fig. 1.

Case d=1.— Now consider the quantum-mechanical Hamiltonian

$$H=-\tfrac{1}{2}d^2/dx^2+\tfrac{1}{2}M^2x^2(x^2M)^\delta. \tag{12}$$

For this Hamiltonian we have calculated the series in (2) corresponding to the ground-state energy $E_0(\delta)$ to order δ^2:

$$E_0(\delta)=\tfrac{1}{2}M+\tfrac{1}{4}\delta M\psi(\tfrac{3}{2})+\tfrac{1}{128}\delta^2 M[-\psi''(\tfrac{3}{2})-8\psi'(\tfrac{3}{2})\ln2+8\psi^2(\tfrac{3}{2})-16\psi(\tfrac{3}{2})+32-32\ln2]+O(\delta^3). \tag{13}$$

Note that the structure of this series is very similar to that of the series in (11). We have calculated the series in (13) in two independent ways. First, we used the recipe described in this paper; second, we used standard Rayleigh-Schrödinger perturbation theory. The series gives excellent numerical results when $|\delta|<1$.

Large-N approximation.— We treat the Lagrangean L in the large-N approximation by replacing ϕ with the N-component field $\mathbf{\Phi}$; we then consider the new Lagrangean

$$L_N=\tfrac{1}{2}(\partial\mathbf{\Phi})^2+\tfrac{1}{2}\mu^2\mathbf{\Phi}^2+\lambda M^2\mathbf{\Phi}^2(M^{2-d}\mathbf{\Phi}^2/N)^\delta. \tag{14}$$

In the limit $N\to\infty$ with λ and M^2 fixed, a variational calculation that employs a trial Gaussian ground-state wave functional gives the exact solution to the theory.[8] This solution can also be obtained by a set of graphical rules[4] (summing only "cactus" graphs) which may be derived from saddle-point expansion of the functional-integral representation of the vacuum-vacuum amplitude.

The pole at $p^2=m_R^2$ in the two-point Green's function is exactly calculable in the large-N limit. We compare the results for m_R^2 obtained in two different ways. First, m_R^2 is determined implicitly by our obtaining the exact gap equation in the large-N limit. This gap equation (in which we have set $N=1$) is

$$m_R^2=\mu^2+2\lambda M^2(\delta+1)(I_RM^{2-d})^\delta, \tag{15}$$

where $I_R=\int d^dp(p^2+m_R^2)^{-1}$. Expansion of this implicit solution for m_R^2 as a series in powers of δ gives

$$m_R^2=\mu^2+2\lambda M^2+2\lambda\delta M^2\ln(eI_1)+\delta^2[\lambda M^2\ln(I_1)\ln(e^2I_1)-4\lambda^2M^2I_2\ln(eI_1)/I_1]+\cdots, \tag{16}$$

where

$$I_k=M^{2k-d}\int d^dp(p^2+\mu^2+2\lambda M^2)^{-k}.$$

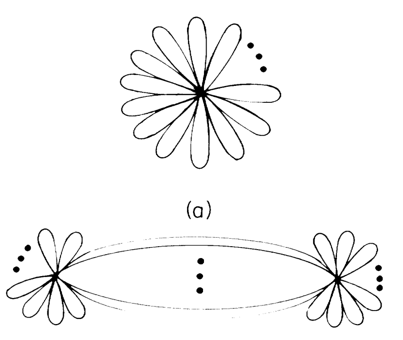

FIG. 1. Graphs contributing to $E(\delta)$ to order δ^2. There are two graphs of type (a): one has vertex 1 and α_1 self-loops, and the other has vertex 2 with α_2 self-loops. There are three classes of diagrams of type (b): one class has two vertices of type 1, one class has two vertices of type 2, and the third class has a type-1 and a type-2 vertex. The graphs of type (b) must be summed on the number of lines joining the vertices. For example, the class of graphs having mixed vertices has $2p$ lines connecting the vertices, α_1-p self-loops on the vertex of type 1, and α_2-p self-loops on the vertex of type 2. The total amplitude is obtained by summing on p from 1 to $\min(\alpha_1,\alpha_2)$. The details of the calculation are given in Ref. 7.

Alternatively, we can use the graphical rules for the $1/N$ approximation to obtain the terms in the δ series in (16). That is, we sum just the "cactus" graphs arising from the large-N generalization of (4) and apply the derivative operator D in (5). We have verified so far that the resulting series in δ is identical to that in (16) to order δ^3. (The detailed calculations for the two-, four-, and six-point Green's functions are given in Ref. 7.) Thus the process of expanding in powers of δ commutes with the $1/N$ approximation (and also commutes with the process of expressing the two-point function in terms of one-particle graphs).

Note also that the terms in the series in (16) are much less divergent than those in the unrenormalized weak-coupling expansion: Let Λ be the large-moment cutoff. In the usual weak-coupling series there are terms that diverge like Λ^2 and $\ln\Lambda$ when $d=4$. As one can see, in the δ series the corresponding terms diverge like $\ln\Lambda$ and $\ln(\ln\Lambda)$. We believe that the δ series in (2) for the Green's functions is renormalizable, order by order in powers of δ. The renormalization procedure is discussed in Ref. 7.

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[1]For example, in a simple quantum-mechanical system with tunneling, when tunneling occurs rapidly because of a low barrier potential, the Wentzel-Kramers-Brillouin (WKB) method gives a very poor approximation to the tunneling amplitude.

[2]For example, in electrodynamics $g-2$ is an unknown, but surely complicated, function of α. Its weak-coupling expansion $g-2=c_1\alpha+c_2\alpha^2+\cdots$ only makes sense in the limit $\alpha\rightarrow 0$. This expansion does not even begin to suggest how $g-2$ depends on the parameter α when α is not small.

[3]L. Yaffe, Rev. Mod. Phys. **54**, 407 (1982), and references therein; L. D. Mlodinow and N. Papanicolaou, Ann. Phys. **128**, 314 (1980), and **131**, 1 (1980); C. M. Bender, L. D. Mlodinow, and N. Papanicolaou, Phys. Rev. A **25**, 1305 (1982).

[4]K. Wilson, Phys. Rev. D **7**, 2911 (1973). For a review, see S. Coleman, in *Pointlike Structures Inside and Outside of Hadrons,* edited by A. Zichichi (Plenum, New York, 1982).

[5]G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); G. Veneziano, Nucl. Phys. **B117**, 519 (1976); G. C. Rossi and G. Veneziano, Nucl. Phys. **B123**, 507 (1977); E. Witten, in *Quarks and Leptons,* edited by M. Levy and J.-L. Basevant (Plenum, New York, 1980).

[6]An alternative possibility would be to consider a self-interaction term of the form $\lambda\phi^{2/\delta}$ to seek an expansion in δ for $\delta\ll 1$. The first term in such a series is the solution to a free-field theory confined to the infinite-dimensional square well $|\phi^2|<1$. However, this approach has many disadvantages. First, even the leading term in this series is extremely difficult to compute. Second, the form of the series is quite complex; there are terms like $\delta^a(\ln\delta)^b$ in the expansion. Finally, in one-dimensional space-time (quantum mechanics) quantities such as the ground-state energy $E_0(\delta)$ are rapidly varying functions of δ near $\delta=0$. Thus a large number of terms in the δ series would be required for good numerical accuracy.

[7]C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., to be published.

[8]W. A. Bardeen and M. Moshe, Phys. Rev. D **28**, 1372 (1983), and **34**, 1229 (1986).