SU(2) Spectroscopy in Intermediate Volumes

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We present results for the low-lying energies of pure SU(2) gauge theory in a cubic volume up to a size of five glueball masses, based on an analytic-variational continuum calculation. We observe as lowest mass a $T_2(2^+)$ state, roughly half the mass of the almost degenerate $A_1(0^+)$ and $E(2^+)$ states. The $A_1(0^-)$ state shows a pronounced volume dependence. There is fair agreement with existing finite-volume Monte Carlo results.

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This Letter will give the results of our calculations for the low-lying energy spectrum of pure SU(2) gauge theory in a finite cubic volume. It goes beyond the earlier perturbative[1] and semiclassical calculations[2] and overlaps fully with existing Monte Carlo data based on (adjoint) Polyakov loop correlations.[3] Our main motivation was and still is to *understand* confinement, expecting this to have a purely gluonic origin. The challenge is to use only the Yang-Mills Lagrangean as input, without any free parameters except for the mass scale. We believe that our present calculations bring us close to the point where confinement (string formation) sets in, and provide the proper tools for the calculation of glueball masses and string tensions and for the unlocking of the "secrets" of confinement.

Nevertheless, of more immediate use will be the comparison of our results with Monte Carlo calculations, which is why we present these results in this form and defer technical details to a future publication. Realizing our responsibility in establishing a reference frame, we have taken great care to demonstrate stability of our results. We base these results on a Rayleigh-Ritz calculation, where we provide both upper and lower bounds on the energy. We also include a two-loop correction. To facilitate the comparison of our results and the lattice Monte Carlo data in a finite volume, we convert to dimensionless ratios (except in Fig. 2, curves c). As discussed elsewhere,[4] this removes the ambiguity due to our lack of precise information on the nonperturbative β function.[5] We consider the parameters z_R and $\mathcal{E}_R^{(i)}$, being respectively the linear dimension L of the cubic volume and the energy of 't Hooft-type electric flux[6] in units of some mass associated to an irreducible representation R of the cubic group. This group has the representations[7] $R=\{A_1,A_2,E,T_1,T_2\}$ with dimensions $d_R=\{1,1,2,3,3\}$ occurring with both parities. The lowest angular momentum these representations couple to are $\{0,3,2,\ 1,2\}$, respectively. The electric flux is classified by the vector $\mathbf{e}\in Z_2^3$ and using the cubic symmetry we get three different energies, determined by the number of components of $\mathbf{e}$ equal to 1 (mod 2) (we call this the number of "units" of electric flux, i.e., $\mathcal{E}^{(i)}$ is the energy of i units of electric flux). Lattice Monte Carlo data for the volumes where we expect our result to be accurate is available in the form of z_{E^+}, $(\mathcal{E}_E^{(1)}/z_{E^+})^{1/2}$ and $z_{A_1^+}/z_{E^+}$.[3] In the Monte Carlo calculations energy differences are measured by use of time-time correlations for spatial Polyakov loops on elongated lattices (adjoint loops for "glueballs" and fundamental loops for "string tensions," the quotation marks are explained in Ref. 4).

We first briefly outline our approach, which is based almost entirely on Lüscher's effective Hamiltonian for the spatially constant gauge fields,[1] with, however, the crucial addition of proper boundary conditions. These boundary conditions reflect the nontrivial topology of configuration space (ignoring this leads to a Gribov ambiguity[8]), and is also related to the fact that a zero-momentum configuration represented by a spatially constant gauge potential is not a gauge-invariant notion. The boundary conditions are the remnant, in an adiabatic approximation, of the patching of coordinate systems in configuration space. This adiabatic approximation can be explicitly tested in our calculational scheme and will be discussed in more detail elsewhere. The coordinates of Lüscher's effective Hamiltonian are the spatially constant vector potentials $c^a{}_i$ with a the SU(2) color indices and i the space indices. The symmetries are given by $c^a{}_i\to\xi^{ab}c^b{}_j\eta_{ij}$, with $\xi\in\mathrm{SO}(3)$ and $\eta\in\mathrm{O}(3,Z)$. The effective Hamiltonian is given in terms of the minimally subtracted renormalized coupling constant g at the scale L by

$$LH_{\mathrm{eff}}=-\tfrac{1}{2}(1/g^2+\alpha_1)^{-1}\partial^2/\partial c^a{}_i{}^2+V_T(c)+V_1(c)+V_2(c)+\cdots. \tag{1}$$

Here V_T is the transverse potential vanishing along Abelian configurations, called the vacuum valley,[9] while V_1 and V_2

are the one- and two-loop contributions to the effective potential along the vacuum valley,

$$V_1(c)=\frac{4}{\pi^2}\sum_{\mathbf{n}\neq\mathbf{0}}\frac{\sin^2(\mathbf{n}\cdot\mathbf{r}/2)}{(\mathbf{n}^2)^2}-2|\mathbf{r}|,\quad r_i=\left(\sum_{a=1}^{3}c^a{}_i c^a{}_i\right)^{1/2}, \tag{2a}$$

$$V_2(c)=\frac{g^2}{32}\{[\Delta V_1(c)]^2+2\Delta V_1(c)\Delta V_1(0)\},\quad \Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial r_i^2}, \tag{2b}$$

$$V_T(c)=\frac{1}{4}\left(\frac{1}{g^2}+\alpha_2\right)\sum_{ija}F^a{}_{ij}F^a{}_{ij}+\alpha_3\sum_{i,j,k,a,b}F^a{}_{ij}F^a{}_{ij}c^b{}_k c^b{}_k+\alpha_4\sum_{i,j,a,b}F^a{}_{ij}F^a{}_{ij}c^b{}_j c^b{}_j+\alpha_5[\det(c)]^2+\cdots, \tag{2c}$$

where $\alpha_1=2.1810429\times10^{-2}$, $\alpha_2=7.5714590\times10^{-3}$, $\alpha_3=-1.1130266\times10^{-4}$, $\alpha_4=-2.1475176\times10^{-4}$, $\alpha_5=-1.2775652\times10^{-3}$, and $F^a{}_{ij}=\epsilon_{abd}c^b{}_i c^d{}_j$. We have extended Lüscher's effective Hamiltonian to higher orders relevant for our nonperturbative analysis. The need for the one-loop term in Eq. (2a) was discussed previously.[2] The terms proportional to α_3, α_4, and α_5 in Eq. (2c) were obtained from a one-loop calculation and the two-loop term of Eq. (2b) follows from a straightforward extension of previous two-loop calculations for chromomagnetic vacuum energies.[10] We made a polynomial fit to eighth order in r_i for $V_1(c)$ and $V_2(c)$, accurate to one-third of 1% for r_i smaller than π. Terms of sixth order are the appropriate Taylor coefficients, whereas the eighth-order term is considered as a correction. Finally the boundary conditions are given by

$$\left(\frac{\partial}{\partial r_i}\right)^{1-e_i} r_i\Psi_{\mathbf{e}}(c)\,|_{r_i=\pi}=0. \tag{3}$$

We use spherical coordinates (r_i,θ_i,ϕ_i) for the vectors $(c^1{}_i,c^2{}_i,c^3{}_i)$, and the problem becomes that of three interacting particles enclosed in a sphere. Gauge invariance requires the total "angular momentum" to be zero.

Next we analyze this Hamiltonian using the Rayleigh-Ritz technique and in general we take a basis of the form

$$\langle c\,|\,l_1l_2l_3n_1n_2n_3\rangle=\sum_m W(l_1l_2l_3m_1m_2m_3)\prod_i\chi_{l_in_i}(r_i)Y_{l_im_i}(\theta_i,\phi_i). \tag{4}$$

Here W are the Clebsch-Gordan-Wigner coefficients for combining three SU(2) irreducible representations into a (color) singlet, thereby resolving the problem of eliminating the gauge degrees of freedom; each element of our basis is gauge invariant. The $\chi_{l_in_i}(r_i)$ are one-particle radial eigenfunctions and satisfy the boundary conditions $(d/dr_i)^{1-e_i}r_i\chi_{l_in_i}(r_i)\,|_{r_i=\pi}=0$. For simplicity we chose $\chi_{l_in_i}(r_i)=j_{l_i}(w_{n_i}^{(e_i)}r_i)$. These spherical Bessel functions allow us to calculate the Hamiltonian [Eq. (1)] with respect to the basis in Eq. (4) *algebraically* in the "momenta" $w_{n_i}^{(e_i)}$. The latter are easily calculated to a high precision. Since we expect the wave function to become less peaked for large g, this should be a good basis for that region. Furthermore, we restricted the basis by suitable projections to the various representations of the cubic group (or whatever symmetry is left in the presence of nonzero electric flux), and most states considered are ground states in these particular sectors. Typically we took 350–450 basis vectors, chosen such that they contain the most relevant coefficients of the wave function, and diagonalized the Hamiltonian using the IMSL (International Mathematical Subroutine Library) routines. This provides an upper bound on the energies. To make absolutely sure our results were accurate we also calculated a lower bound, using Temple's inequality.[11] (It requires one to calculate in addition the matrix of H^2.)

Figure 1 gives for $g\geq0.8$ our results for the low-lying energy values (in units of $1/L$, L being the physical size of the cubic volume). States are labeled by their representation [e_i^+ denotes the (ground) state with i units of electric flux]. Energies were calculated for steps of 0.2 in g. The full lines result from the Hamiltonian [Eq. (1)], excluding the two-loop contribution, the terms of order 6 in the transverse part of the potential (V_T) and the terms of order 8 in the one-loop contribution (V_1).

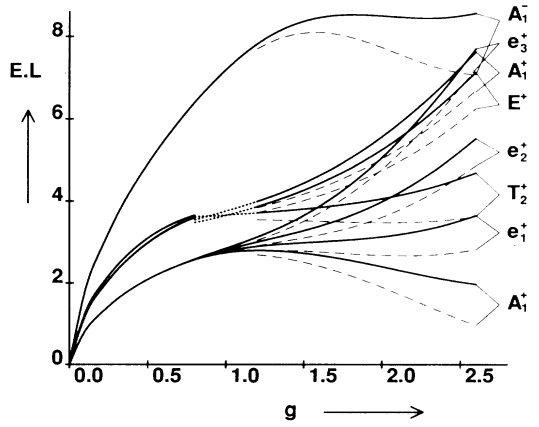

FIG. 1. The energies in units of $1/L$, labeled by their representation, for a few low-lying levels as a function of the renormalized coupling constant. The dotted parts of the curves use only the upper bound. Elsewhere upper and lower bound coincide within the thickness of the curves. Below $g=0.8$ we show the perturbative result (Ref. 1). The dashed curves give the results including the higher-order corrections.

The upper and lower bounds coincide within the thickness of these lines. Where the lower bound becomes less accurate, but we still expect the upper bound to be reasonable, we dotted the curve, using the value of the upper bound. The full lines for $g \leq 0.8$ gives the perturbative result of Lüscher and Münster.[1] Tunneling for the A_1^+ excited and the E^+ ground states is expected to set in around $g=0.5$, which explains the reasonably large deviation for these states, but otherwise the agreement is impressive. The dashed curves give the result for the full effective Hamiltonian [Eq. (1)]. The eighth-order one-loop term, the sixth-order transverse part and the two-loop term contribute approximately with the ratios 0.1:2:3.

We observe the remarkable fact that the ground-state energy decreases beyond $g \sim 1.3$, corresponding to $z_{E^+} \sim 1.5$. This might indicate an instability of the vacuum under formation of magnetically neutral domains, which was speculated on in Ref. 10. We believe this to be a significant step towards understanding the long-range behavior of the Yang-Mills vacuum and certainly deserves more attention than we can give it here.

In Fig. 2, curves *a* and *b*, we compare our results for the string tension $(\mathcal{E}_E^{(1)}/z_{E^+})^{1/2}$ and the mass ratio $z_{A_1^+}/z_{E^+}$ as functions of z_{E^+} with existing Monte Carlo results.[3] Figure 2 is obtained as follows from Fig. 1: $z_{E^+}=L(E_{E^+}^{(0)}-E_{A_1^+}^{(0)})$, $z_{e_1^+}=L(E_{e_1^+}^{(0)}-E_{A_1^+}^{(0)})$, $z_{A_1^+}=L(E_{A_1^+}^{(1)}-E_{A_1^+}^{(0)})$, and $\mathcal{E}_{E^+}^{(1)}=z_{e_1^+}/z_{E^+}$. We see that the results agree reasonably well. (For the string tension, compare also Fig. 2 of the first paper in Ref. 2, but note the E/A_1 mixing.[3,12]) The mass ratio in our calculations is to good accuracy identically 1.1, whereas the Monte Carlo data are somewhat higher for z_{E^+} between 1.5 and 3.5. The systematic errors in our method are the higher-order corrections in g and the nonadiabatic corrections, whereas in the Monte Carlo method the time scales used might not be large enough compared with the relevant energy differences.[4,12] Furthermore, the lattice approximation (the finite a correction) will also lead to systematic errors (nonuniversality). Initial expectations[2] were that our result would only be reliable up to about $z_{E^+}=2$, but surprisingly z and $\mathcal{E}$ are stable within (2–4)% under the higher-order corrections discussed previously, all the way to $z_{E^+}=5$.

From Fig. 1 one deduces that z_{E^+} and $z_{A_1^+}$ are almost constant functions of g around $g=1$, which is largely due to the substantial lowering of the excited energies due to the tunneling phenomenon. It is therefore not permissible to convert an expansion in g to an expansion in z, as was done both in Refs. 1 and 2. But it does explain the sharp onset of tunneling in Fig. 2, curves *b*, at $z_{E^+} \sim 0.9$ and the reasonable accuracy of our semiclassical prediction[2] for this onset ($z_{A_1^+} \sim 1.0$).

In Fig. 2, curves *c*, we convert our values for $z_{E^+}(g)$ to the physical mass by

$$M(E(2^+)) = z_{E^+}(g)(11g^2/24\pi^2)^{51/121}\exp(12\pi^2/11g^2)\Lambda_{\overline{\mathrm{MS}}},$$

where MS denotes the minimal-subtraction scheme (the three-loop result of Tarasov, Vladimirov, and Yu[13] suggests an error of 6% or less), giving a plateau[1,14] at $M(E(2^+)) \sim 16\Lambda_{\overline{\mathrm{MS}}} \sim 119\Lambda_L$, which is below the Monte Carlo result.[12] This is probably due to higher-order corrections in the β function.[5]

Finally Fig. 3 gives some new predictions. First, a $T_2(2^+)$ mass surprisingly lower than the $E(2^+)$ and the $A_1(0^+)$ mass. Second, a pronounced z dependence of the $A_1(0^-)$ mass. Still, we do *not* consider it prudent to predict the lowest glueball state to have $J^P=2^+$ at large z. Only when we understand how rotational invariance is restored at large z will we be able to tell *which* E^+ doublet combines with this T_2^+ triplet state into a $J^P=2^+$ quintet,[1] and what its final mass will be. More

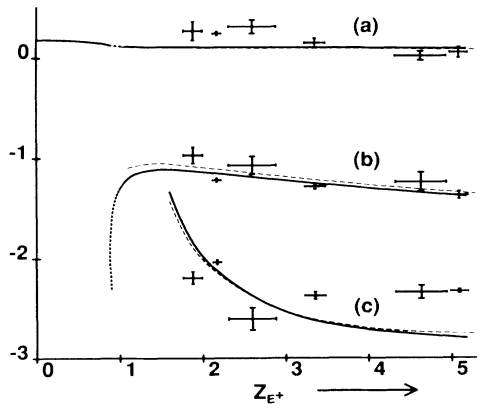

FIG. 2. Our results compared to lattice Monte Carlo results (Refs. 3 and 12) for (curves *a*) $\ln(z_{A_1^+}/z_{E^+})$, (curves *b*) $\ln(\mathcal{E}_E^{(1)}/z_{E^+})^{1/2}$, and (curves *c*) $\ln[M(E(2^+))/(250\Lambda_{\overline{\mathrm{MS}}})]$. The conventions for the curves are as in Fig. 1.

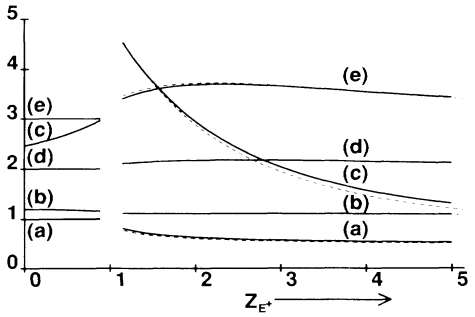

FIG. 3. Results for the following mass ratios: (curves *a*) $z_{T_2^+}/z_{E^+}$, (curves *b*) $z_{A_1^+}/z_{E^+}$, (curves *c*) $z_{A_1^-}/z_{E^+}$, and ratios of electric-flux energies: (curves *d*) $\mathcal{E}^{(2)}/\mathcal{E}^{(1)}$, (curves *e*) $\mathcal{E}^{(3)}/\mathcal{E}^{(1)}$. The conventions for the curves are as in Fig. 1.

accurately measurable with lattice Monte Carlo techniques, we believe, is our prediction for the energy ratios of different amounts of electric flux. For two (three) units of electric flux we propose the use of two (three) spatial Polyakov loops, in different directions. The time-time correlation of such an operator should then allow one to estimate the energy of two (three) units of electric flux. These ratios can be an important tool for testing the string picture, because such a picture predicts[6] $\mathcal{E}^{(i)}/\mathcal{E}^{(1)} = z_{e_i^+}/z_{e_1^+} = \sqrt{i}$, which deviates considerably from the values which we find below $z_{E^+} = 5$. For the lattice we suggest plotting this ratio as a function of $\sqrt{z_{e_1}}$, because z_{e_1} can be measured more accurately than z_E. Since there are various indications[2,4] that string formation sets in for $z_{E^+} \geq 5$, it would be important to see if and how quickly these ratios of electric-flux energies settle to their expected asymptotic values.

In conclusion, we feel confident that we have demonstrated the nature of the nonperturbative dynamics of SU(2) gauge theory in a finite but reasonably large volume. Of course, the physically interesting domain is still beyond the distances that we have probed, but we are getting close.

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