Solvable Model of the Beam-Beam Limit in e^+e^- Colliding Rings

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For the beam-beam phenomenon in e^+e^- colliding rings, a simple model is presented, which illustrates several common features of the observed phenomenon (a blowup of one beam above a certain current, the unavoidable beam-beam limit, etc.). The tune-shift limit is given as a function only of betatron tune and radiation damping rate per collision point. This implies the universality of the phenomenon.

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In all high-energy e^+e^- colliding storage rings, it is commonly observed[1] that although the luminosity L increases as I^2 (I being beam current) for small I, L becomes proportional only to I when I exceeds a certain critical value. This is equivalent to the saturation of the beam-beam parameter[1] (ξ). Remarkably,[2] the saturated value (ξ_∞) is almost universally 0.05, although the appearances of the phenomenon are quite complicated.

Since the achievable L is limited by ξ_∞, understanding of the phenomenon has been one of the most important problems. In particular, the saturation mechanism of ξ is little understood, though it is clearly seen in a numerical multiparticle tracking[3] (MPT). It is almost certain[1] that the beam height increases in proportion to I; there seems, however, to be no theoretical explanation of it.

We will present a simple model that explains the saturation and related universal phenomena. Let us consider a ring with N_s interaction points (IP's) and $N_s/2$ bunches of equal intensity in each beam. To make our model simple, we use a flat-beam[4] limit (FBL), where the horizontal motion can reasonably be assumed to be unaffected, and assume that coherent dipole motion is stable and ignorable. The next plausible assumption is that the particle distribution giving the beam-beam force at an IP can be treated as Gaussian.

We have now only to study the motion of a particle in a bunch in the vertical direction y. The motion can be described by successive operations of the following three mappings: **O** (oscillation),

$$\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \hat{U} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix};$$

B (beam-beam force),

$$Y_1' = Y_1 \text{ and } Y_2' = Y_2 + G(Y_1),$$

$$G(Y) = -\kappa \operatorname{erf}[Y/(2\Lambda_{11}^*)^{1/2}];$$

R (radiation),

$$Y_1' = Y_1 \text{ and } Y_2' = \lambda Y_2 + \{(1-\lambda^2)\varepsilon_y\}^{1/2}\hat{r}.$$

Here $Y_1 = y/\sqrt{\beta_y}$ and $Y_2 = (\alpha_y y + \beta_y y')/\sqrt{\beta_y}$ are the canonical variables defined in terms of nominal Twiss parameters, Λ_{11}^* is the average of Y_1^2 of the partner bunch under collision, μ ($=2\pi\nu$) is the betatron phase advance per IP, λ is the damping ratio defined by $\lambda = \exp(-2\delta)$ with δ being the damping decrement[5] [(revolution time)/(betatron damping time per IP)], ε_y is the nominal vertical emittance, and $\hat{r}$ is a noise with unit standard deviation. Lastly $\kappa \equiv 2\pi^{3/2}\sqrt{\varepsilon_y}\,\eta$ defines the vertical beam-beam force averaged over the horizontal direction, where η is the nominal beam-beam parameter,

$$\eta = \frac{N r_e}{2\pi\gamma}\left(\frac{\beta_y^0}{\beta_x^0}\right)^{1/2} \frac{1}{(\varepsilon_x \varepsilon_y)^{1/2}}.$$

Here N is the number of particles in the partner bunch, r_e the classical electron radius, γ the relativistic Lorentz factor, $\beta_{x,y}^0$ the nominal β function at an IP, and ε_x the nominal horizontal emittance. We have treated the effect of radiation as if it works locally. Since essentially the effect belongs to linear dynamics, this does not lead to unphysical results.

Most previous theoretical works[6] studied the mapping **BO** only and assumed that the partner bunch is strong and unaffected. This picture, however, does not seem to illustrate the essential features of the phenomenon, since (i) the presence of strong fluctuations and damping (**R**) considerably perturbs the Hamiltonian dynamics of the **BO** system, (ii) as asserted by Chao,[2] the overlapping resonances do not produce a stochastic region leading to infinity for our case and the gross beam behavior does not depend sensitively on whether the stochastic limit has been exceeded or not, and (iii) even in a most detailed and successful case,[7] including synchrotron-betatron coupling, the predicted stochastic limit gives too large ξ_∞ for proton rings.

Since the gross beam behavior is the first thing to be studied, we, in this paper, are not concerned with a single-particle incoherent motion: We had better study statistical quantities such as $\Lambda_{ij}^{(\pm)} \equiv \langle Y_i Y_j \rangle^{(\pm)}$, where $\langle\,\rangle^{(\pm)}$ is an average over an $e^\pm$ bunch. Then under **B**, $\Lambda_{ij}^{(\pm)}$ experiences a variation, $\Lambda_{11}^{(\pm)\prime} = \Lambda_{11}^{(\pm)}$, $\Lambda_{12}^{(\pm)\prime} = \Lambda_{12}^{(\pm)} + \langle Y_1 G(Y_1)\rangle^{(\pm)}$, and $\Lambda_{22}^{(\pm)\prime} = \Lambda_{22}^{(\pm)} + 2\langle Y_2$

$\times G(Y_1)\rangle^{(\pm)}+\langle G(Y_1)^2\rangle^{(\pm)}$. Note that the averages should be evaluated with the distribution just before **B**. To do this, we must know the phase-space distribution function $\psi(Y_1,Y_2)$.

As an empirically permissible assumption, we approximate ψ always as Gaussian:

$$\psi(Y_1,Y_2)=\frac{1}{2\pi(\det\Lambda)^{1/2}}\exp\left[-\frac{1}{2}\sum_{i,j}\Lambda_{ij}^{-1}Y_iY_j\right],$$

ignoring any finer details of ψ. It brings some unphysical feature to our model (see discussion given later); instead of this, we can obtain a deterministic algebraic mapping system:

$$\mathbf{O}',\ \Lambda_{ij}^{(\pm)\prime}=(\hat{U}\Lambda^{(\pm)}\hat{U}^{-1})_{ij},$$

$$\mathbf{B}',\ \Lambda_{11}^{(\pm)\prime}=\Lambda_{11}^{(\pm)},\ \Lambda_{12}^{(\pm)\prime}=\Lambda_{12}^{(\pm)}+2\kappa A(R^{\pm1})(\Lambda_{11}^{(\pm)})^{1/2},\ \Lambda_{22}^{(\pm)\prime}=\Lambda_{22}^{(\pm)}+4\kappa A(R^{\pm1})\frac{\Lambda_{12}^{(\pm)}}{(\Lambda_{11}^{(\pm)})^{1/2}}+4\kappa^2B(R^{\pm1}),$$

$$\mathbf{R}',\ \Lambda_{11}^{(\pm)\prime}=\Lambda_{11}^{(\pm)},\ \Lambda_{12}^{(\pm)\prime}=\lambda\Lambda_{12}^{(\pm)},\ \Lambda_{22}^{(\pm)\prime}=\lambda^2\Lambda_{22}^{(\pm)}+(1-\lambda^2)\varepsilon_y,$$

where $R=\Lambda_{11}^{(-)}/\Lambda_{11}^{(+)}$, $A(R)=-1/\{2\pi(1+R)\}^{1/2}$, and $B(R)=\frac{1}{4}-\arcsin\{[R/2(1+R)]^{1/2}\}/\pi$.

Let us track the behavior of Λ_{ij}. The mapping $\mathbf{R}'\mathbf{B}'\mathbf{O}'$ turns out to have a period-one fixed point $\tilde{\Lambda}_{ij}$ on a Poincaré surface of section built just before $\mathbf{B}'$:

$$\tilde{\Lambda}_{11}^{(\pm)}=\kappa^2[-D(\tilde{R}^{\pm1})+\{\varepsilon_y\kappa^{-2}+D(\tilde{R}^{\pm1})^2+E(\tilde{R}^{\pm1})\}^{1/2}]^2,$$

$$\tilde{\Lambda}_{12}^{(\pm)}=-\frac{2\lambda}{1+\lambda}\kappa A(\tilde{R}^{\pm1})(\tilde{\Lambda}_{11}^{(\pm)})^{1/2},\ \tilde{\Lambda}_{22}^{(\pm)}=\varepsilon_y+\kappa^2E(\tilde{R}^{\pm1}),$$

where

$$D(R)=-\frac{2\lambda}{1+\lambda}\frac{1}{\tan\mu}A(R),\ E(R)=\frac{4\lambda^2}{1-\lambda^2}\left\{B(R)-\frac{2\lambda}{1+\lambda}A(R)^2\right\},$$

and $\tilde{R}$ is the root of the equation $R=h(R)$:

$$h(R)=\frac{[-D(R^{-1})+\{\varepsilon_y\kappa^{-2}+D(R^{-1})^2+E(R^{-1})\}^{1/2}]^2}{[-D(R)+\{\varepsilon_y\kappa^{-2}+D(R)^2+E(R)\}^{1/2}]^2}. \quad (1)$$

Derivation of $\tilde{\Lambda}$ is easy when we notice the changes of $\det\Lambda$, $\mathrm{tr}\Lambda$, and Λ_{11} under the mappings.[8]

Let us here define strictly the (perturbed) beam-beam parameter ξ as $\xi=\{2\varepsilon_y/(\Lambda_{11}^{(+)}+\Lambda_{11}^{(-)})\}^{1/2}\eta$. Then $L=L_0\xi/\eta$, where L_0 is the nominal value of L: $L_0=N^2f_0N_s/8\pi(\beta_x^0\beta_y^0\varepsilon_x\varepsilon_y)^{1/2}$ with f_0 the revolution frequency.

Returning to Eq. (1), $\tilde{R}$ is an implicit function only of λ, μ, and η. It is easily seen that it has trivial solutions 0, 1, and ∞, and when R is a root, so is R^{-1}. We find numerically that $\tilde{R}=1$ is the only stable solution for small η, whereas a bifurcation occurs and a couple of solutions is born at a certain point η_b. In Fig. 1, $\tilde{R}$ is shown as a function of η with use of model values of TRISTAN's main ring [ν=0.05 and 0.2 (mod 1/2), $\delta=7\times10^{-4}$] as an example. For these model values, $\eta_b=0.0257$. In our model, the $\tilde{R}$-η curve (and η_b) is surprisingly independent of ν; the curve in Fig. 1 is almost the same for all ν. This seems to be a special consequence of FBL.

We can also numerically show that $\tilde{R}=1$ is unstable if $\eta>\eta_b$, whereas $\tilde{R}\neq1$ are stable there. The beams prefer such an asymmetric state. It is also shown in Fig. 1 that one of the beams is blown up rapidly as soon as η exceeds η_b, which shortens the lifetime of the beam, or, at least, reduces L and ξ considerably as shown by a solid line in Fig. 2. The sudden decrease of ξ is actually seen[9] and may limit L for some rings.

It may be expected, however, that such an asymmetric state can be avoided, to some extent, because of effects not considered here. In fact, the asymmetry is controlled by an ingenious empirical method[10]; and MPT[3] shows that the state $\tilde{R}=1$ can be maintained by a fine selection of ν.

It will thus be of interest to see what happens when $\tilde{R}=1$ is somehow maintained. In this case, ξ is a simple

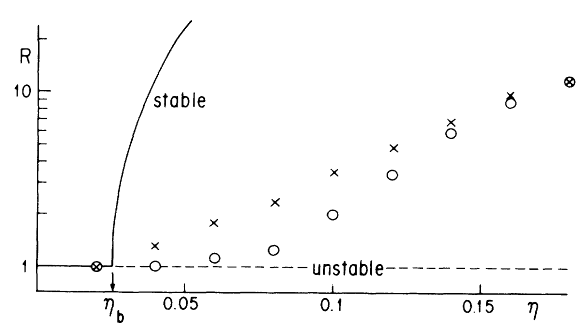

FIG. 1. Roots of $R=h(R)$ as functions of η (upper half), and results of MPT for ν=0.05 (circles) and 0.2 (crosses).

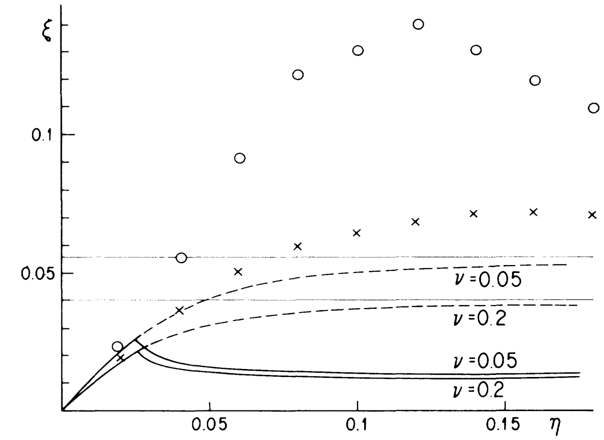

FIG. 2. Perturbed beam-beam parameter ξ as a function of η. Dashed lines correspond to the case when the bifurcation is avoided. Results of MPT are also given for $\nu=0.05$ (circles) and 0.2 (crosses).

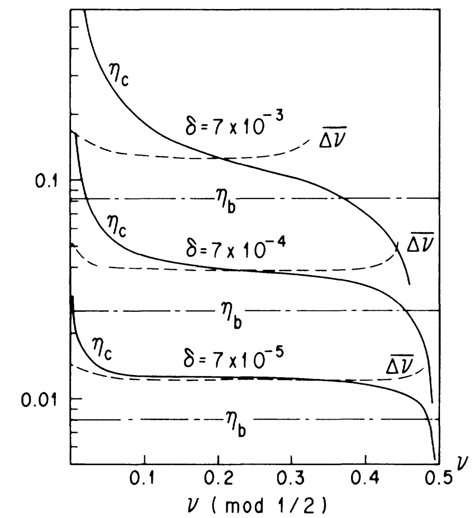

FIG. 3. Dependences of η_b, η_c, and $\overline{\Delta\nu}$ on ν and δ.

function of η, μ, and λ:

$$\xi=(2\pi^{3/2}\{-D(1)+[\varepsilon_y\kappa^{-2}+D(1)^2+E(1)]^{1/2}\})^{-1}, \tag{2}$$

which is shown by dashed lines in Fig. 2. From Eq. (2) and the figure, the saturation of ξ for large η is clearly seen. The line is quite similar to that given in Ref. 3. The saturated value is

$$\xi_\infty=(2\pi^{3/2}\{-D(1)+[D(1)^2+E(1)]^{1/2}\})^{-1}.$$

We define the critical η, η_c, by $\eta_c\equiv\xi_\infty$ as the turning point from $L\propto I^2$ to $L\propto I$. For our model values, $\eta_c=0.055$ (0.04) for $\nu=0.05$ (0.2). The tune-shift limit $\overline{\Delta\nu}$ is defined by $\cos(\mu+2\pi\overline{\Delta\nu})=\cos\mu-2\pi\eta_c\sin\mu$, which is 0.040 (0.039) for $\nu=0.05$ (0.2) for our model values.

We show η_b,η_c, and $\overline{\Delta\nu}$ as functions of ν for some δ's in Fig. 3. Since $D(1)$ becomes ∞ at $\nu\to 0^+$, η_c blows up there; this is consistent with experimental data.[11] In the physically meaningful region ($0.05<\nu<0.45$, say), η_c seems almost independent of ν (η_b happens to be almost constant). Further the δ dependence of η_c is quite weak; when δ is small enough, η_c is dominated by the factor $1-\lambda^2$ in $E(1)$, which implies $\eta_c\propto\delta^{1/2}\propto\gamma^{1.5}$. This seems consistent with the experimental data given in Ref. 1. In this respect, the universality of $\eta_c(=\xi_\infty)$ can be explained as a result of the fact that in every high-energy e^+e^- colliding ring, δ has almost the same order of magnitude at its highest energy.

According to Ref. 1, $\overline{\Delta\nu}$ is more universal. This may be related to the fact that $\overline{\Delta\nu}$ is less ν dependent, as shown in Fig. 3. The singularity of η_c at $\nu\to 0^+$ is canceled by that of $\sin\mu$. For our model value, $\overline{\Delta\nu}$ becomes complex for $\nu>0.45$; then, the coherent dipole oscillation will be unstable there.

We have shown, by Gaussian approximation (GA), that the bifurcation occurs even with a complete symmetry between both beams and that the saturation of ξ is inevitable even when the bifurcation is avoided. Now, we can do the same[12] using a round-beam limit, another limiting case. It is interesting to see that the qualitative features are the same also for a round-beam-limit model with GA. The fact that the two oppositely limiting cases lead to the same results implies that we can expect the same for general cases.

It is now interesting to see to what extent our GA model is faithful to the original single-particle **RBO** mapping. MPT based on the mapping is done for fifteen damping times (necessary for the system to reach equilibrium) with 2000 test particles. The results are in good qualitative agreement with our GA results as shown in Figs. 1 and 2. Numerically, in MPT, (i) bifurcation is more smooth and moderate, (ii) ξ is several times as large, and (iii) η_b and η_c are more sensitively dependent on ν, showing resonant behavior for some ν. For some ν, bifurcation is so gentle that the resulting ξ looks as if saturated for some η, although it will decrease eventually. In Fig. 2, $\nu=0.2$ is the case, whereas $\nu=0.05$ shows a sudden decrease of ξ. The results of MPT seem to be in an intermediate stage between the cases represented by the solid line and the dashed line of our model in Fig. 2.

Too small a value of ξ (i.e., too large a value of $\tilde{\Lambda}_{11}$) of our GA model is a natural consequence of the lack of degrees of freedom: only six instead of 2000 of MPT. Under a nonlinear kick such as **B**, Gaussian ψ will produce non-Gaussian fine structure. Notice that the entropy[13] $S\equiv-\langle\ln\psi\rangle$ is const$+(\ln\det\Lambda)/2$ in GA, which increases at **B**, although S should not change under any symplectic mapping such as **B**. Clearly, our system suffers from unphysical heating due to loss of information by GA and $\tilde{\Lambda}_{11}$ has some unphysical additional part. Undoubtedly (and confirmed by MPT), the unphysical part is small for large δ (fast damping) and ∞ for $\delta=0$ (no damping). The unphysical part would be small if we could use more parameters to fit ψ. Our model, thus, cannot be applied to proton rings and electron rings with too small δ.

Related to this is the fact that our GA model cannot incorporate effects that disturb coherence, such as tune spread and nonlinear resonances. This will be the reason for too small values of η_b and too sharp behavior of ξ under the bifurcation and insensitivities of η_b and η_c to tune. With enough parameters, Fig. 3 would have many dips. In this sense, our model yields nonresonant behavior.

Lastly, FBL with vertical motion only is another large simplification. It does not incorporate the synchrotron-betatron and horizontal-vertical[4] couplings.

We have ignored almost all the effects which may lower ξ and L and which may give the phenomenon quite complicated and rich behavior: horizontal and longitudinal motions, aperture limit, nonlinear resonances, non-Gaussian tails, breaking of superperiodicity, and so on. These are, however, secondary factors for the beam-beam-limit phenomena.

The mapping formulation[14] seems to be simple and powerful when the nonlinear force works locally.[13] Our model, though too limited, has shown several common characteristic features of the beam-beam phenomenon. This is the first approximation and gives only a gross feature. To be more precise, it seems reasonable to start with our model and then to include more parameters to incorporate the effects stated above as a next approximation.

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[8]From symmetrical point of view, we had better employ a mapping such as $\boldsymbol{B}/2 \rightarrow \boldsymbol{R}/2 \rightarrow \mathbf{0} \rightarrow \boldsymbol{R}/2 \rightarrow \boldsymbol{B}/2$ ($\boldsymbol{B}/2$ is half of $\mathbf{B}$ and so on). Then $\tilde{\Lambda}_{12}$ would vanish. Since Λ_{11} does not change under $\mathbf{B}$, we prefer the simpler $\mathbf{B} \rightarrow \mathbf{R} \rightarrow \mathbf{0}$ mapping.

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[14]Our mapping method is seemingly similar to that employed by Y. Kamiya and A. W. Chao in Stanford Linear Accelerator Center Report No. SLAC/AP-8, 1980 (unpublished). They ignored $\mathbf{R}$ and linearize $\mathbf{B}$ around natural beam size ($\Lambda_{ij}=\varepsilon_y\delta_{ij}$, for example). Their method is not applicable to the case of large η, where the perturbation on Λ_{ij} becomes large. The best, though a little complicated, way to improve their method (and to extend ours) is to linearize **RBO** mapping around $\tilde{\Lambda}_{ij}$.